Radio-frequency wave simulation in hot magnetized plasma using differential operator for non-local conductivity response

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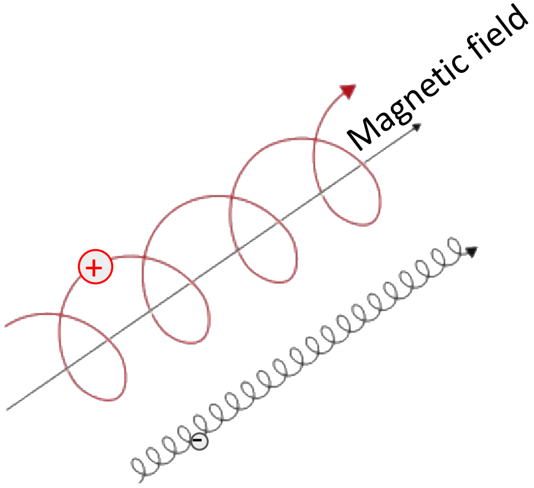
Background : hot plasma has non-local responds RF waves

Plasma

- Charged particles are freely moving (“collisionless”)
- External magnetic field restricts the parlendicuar excursion (“Gyro-motion”)



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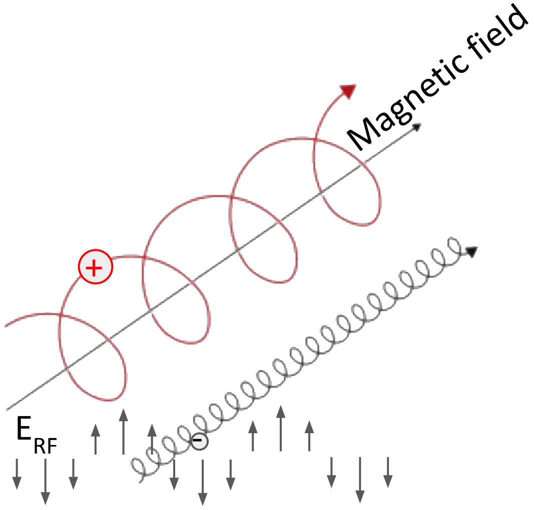

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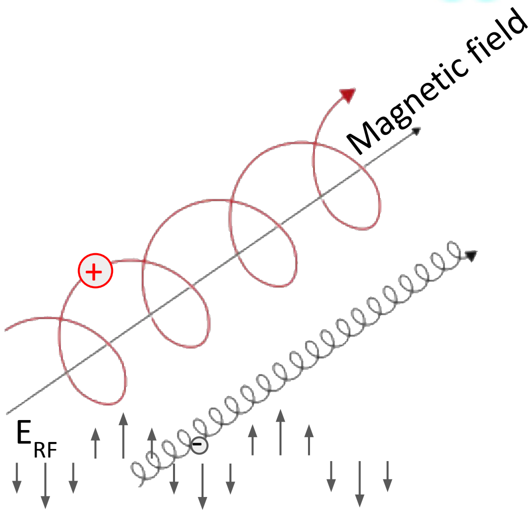


Acceleration from wave fields appears as current in different places (non-local).

Dielectric response is written in wave-number (Fourier) space. In configuration space, the response becomes convolution integral.

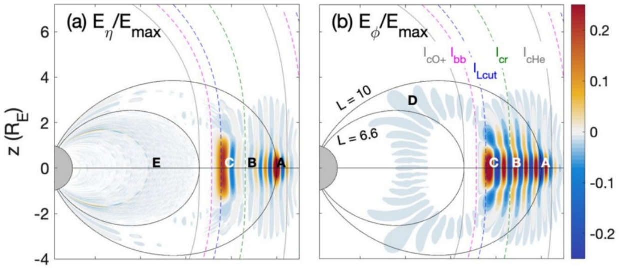
$$\mathbf{J}(\mathbf{k}) = \sigma(\mathbf{k})\mathbf{E}(\mathbf{k})$$

$$\mathbf{J}(x) = \frac{1}{\sqrt{2\pi}} \int dx' \boldsymbol{\sigma}(x - x')\mathbf{E}(x')$$

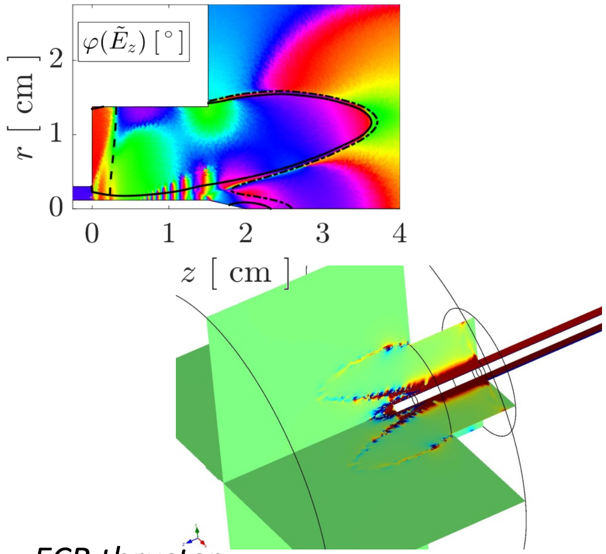
Fourier transformation

Question : How do we incorporate this in numerical full-wave simulations?

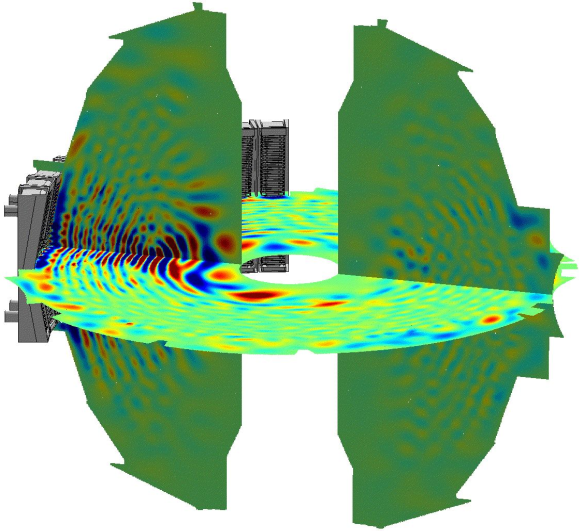
FEM modeling of RF waves in plasma is widely used but majority of them employs local approximation.



EMIC in magnetosphere



ECR thruster



HHFW in NSTX-U spherical tokamak

Freq. domain Maxwell. Eqs.

$$\nabla \times \frac{1}{\mu_0} \nabla \times \mathbf{E} - (\omega^2 \epsilon + i\omega\sigma)\mathbf{E} = i\omega \mathbf{J}_{\mathrm{ext}}$$

depends on x and ω, not k

E. H. Kim, et. al., Geophys. Res. Lett. (2023)
Á. Sánchez-Villar et al, Plasma Source Sci. Tech. (2021)
N. Bertelli, et. al., Nucl. Fusion (2022)

Various approaches to include non-local response exist, but with various limitations

- Full spectral methods:
  - Full-dense matrix, costly to solve

$$\mathbf{E}(x,y) = \sum_{n,m} \mathbf{E}_{n,m} e^{i(k_n x + k_m y)}$$

$$\mathbf{J}_p(x,y) = \sum_{n,m} \sigma(x,y,k_n,k_m)\cdot\mathbf{E}_{n,m} e^{i(k_n x + k_m y)}$$

- Construct a differential operator:

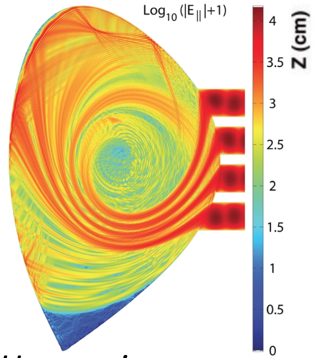
$$\boldsymbol{\sigma}(k_\perp) \simeq \boldsymbol{\sigma}(0) + \boldsymbol{\sigma}' k_\perp + \frac{1}{2}\boldsymbol{\sigma}'' k_\perp + \cdots \qquad k_\perp \Longrightarrow -i\partial/\partial x_\perp$$

  - Typically up to 2nd order: valid only for $k_\perp \rho_i < 1$ [1, 2].
  - Including high order derivative is possible[3] was in 1D.
  - To be precise, derivation is based on kinetic theory.

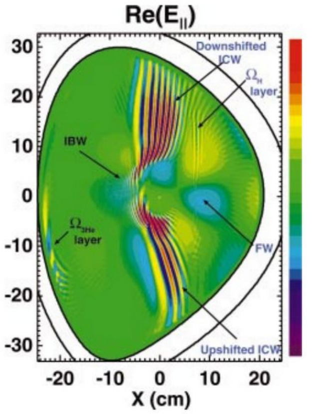
- Iterative addition non-local contribution:
  - ELD for the lower hybrid waves [4, 5].
  - Generalization was not straightforward.

- Convolution integrals:
  - Many publications in 80-90's (in 1D).

Expensive, limited to simpler geometry, not generic enough, and/or does not work well with FEM.



LH wave in C-Mod tokamak[5]



Fast wave MC[2]

[1] M. Brambilla, Plasma Phys. Contr. Fusion (1999)
[2] J. C. Wright et. al., Phys. Plasmas (2004)
[3] D. V. Eester Plasma Phys. Control. Fusion (2013)
[4] O. Meneghini, Phys Plasmas (2009)
[5] S. Shiraiwa, Phys Plasmas (2011)

We could consider this in the context of Green's function

Method of Green's Function

For a given operator L, we want to solve

$$\mathrm{L}\ u(x) = f(x)$$

We look for a function G, which satisfies

$$\mathrm{L}\ G(x, s) = \delta(s - x)$$

Solution can be written as a convolution

$$u(x) = \int G(x, s)\, f(s)\, ds$$

In our case, G is known by back Fourier transforming $\sigma(\mathbf{k})$.

For example, for Maxwellian plasma, the back Fourier transformation in the direction perpendicular to B can be done analytically, and the xx component is

$$\sigma_{xx}(x - x') = \frac{-in_0e^2}{m}\frac{1}{\sqrt{2\pi}}A_n\left[\frac{2n\sqrt{\pi}}{i}\int_0^{2\pi}\frac{d\theta}{2\pi}\frac{|\Omega|}{v_{\mathrm{th}\perp}}\cos\frac{\theta}{2}\exp\left(-\frac{(x-x')^2\frac{\Omega^2}{v_{\mathrm{th}\perp}^2}}{4\sin^2\frac{\theta}{2}} + in\theta\right)\right]$$

Thus, an approach could be….

1) Find L.
2) Solve a coupled PDEs.

$$\nabla \times \frac{1}{\mu_0}\nabla \times \mathbf{E} - \omega^2\epsilon\mathbf{E} + i\omega\mathbf{J}_{hot} = i\omega\mathbf{J}_{\mathrm{ext}}$$

$$L\mathbf{J}_{hot} = f\mathbf{E}$$

In physical space, summed-”screened Poisson potential” fits well the conductivity kernel. In K-space, it corresponds to rational approximation

Using K-space simplifies the operator construction (we focus on the perpendicular direction for now)

$$\mathbf{J}(k_\perp) = \boldsymbol{\sigma}(k_\perp)\mathbf{E}(k_\perp)$$

$$\boldsymbol{\sigma}(k_\perp) \simeq \mathbf{c}_0 + \frac{\mathbf{c}_1}{k_\perp^2 - d_1} + \frac{\mathbf{c}_2}{k_\perp^2 - d_2} + \frac{\mathbf{c}_3}{k_\perp^2 - d_3} \cdots$$

$$\mathbf{J}_1 = \frac{\mathbf{c}_1}{k_\perp^2 - d_1}\mathbf{E}(k_\perp) \Longrightarrow \Delta_\perp \mathbf{J}_1 - d_1 \mathbf{J}_1 = \mathbf{c}_1 \mathbf{E}$$

$$k_\perp \Longrightarrow -i\partial/\partial x_\perp$$

Pros:

- Approximation is valid in all range of $k_\perp$.
- Differential operator is 2nd order.
- No global DoFs coupling (convolution, spectral).

Cons:

- Coupled PDE consisting from E and J.
- Not theoretically derived.

Hints were presented in MFEM community WS in 2022 !?

Non-integer power derivatives (fractional derivatives)

- “a non-integer fractional derivative of f at x = a depends on all values of f , even those far away from a” (Wikipedia)

What is the fractional Laplacian?

Fractional PDEs

Example

$$-\Delta^{\alpha/2} u = 1$$
$$\alpha \in [0,2]$$
$$u(x) = 0 \ \forall x \in \partial\Omega$$

Definition

We follow the *spectral definition* of the fractional laplacian. For regular Laplacian:

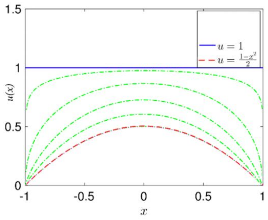
$$-\Delta e_k = \lambda_k e_k \qquad e_k(x) = 0 \forall x \in \partial\Omega$$

$$\Rightarrow -\Delta u(x) = \sum_k \lambda_k (u, e_k)_{L^2_\Omega} e_k$$

For fractional Laplacian:

$$\Rightarrow -\Delta^{\alpha/2} u(x) = \sum_k \lambda_k^{\alpha/2} (u, e_k)_{L^2_\Omega} e_k$$

Intuition



Solution for different fractional exponents.
Blue: $\alpha = 0$
Green (top to bottom): $\alpha \in \{0.1, 0.5, 1.0, 1.5\}$
Red: $\alpha = 2$

Lischke, A., Pang, G., Gulian, M., Song, F., Glusa, C., Zheng, X., Mao, Z., Cai, W., Meerschaert, M. M., Ainsworth, M., & Karniadakis, G. E. (2020). What is the fractional Laplacian? A comparative review with new results. *Journal of Computational Physics*, *404*, 109009. https://doi.org/10.1016/j.jcp.2019.109009

This is the same as the fractional Laplacian ! Except the conductivity is the tensor and plasma parameter changes in space (non-uniform plasma)

Tobias Duswald, et al., 2022 MFEM community workshop

Modified AAA algorithm can be used to approximate exponentially scaled modified bessel using the same pole.

$$\chi_{xx} = \frac{\omega_p^2}{\omega} \sum_{n=-\infty}^{+\infty} \frac{n^2 I_n}{\lambda} e^{-\lambda} A_n$$

In order to handle non-uniform plasma, each Bessel function needs to be approximated separately.

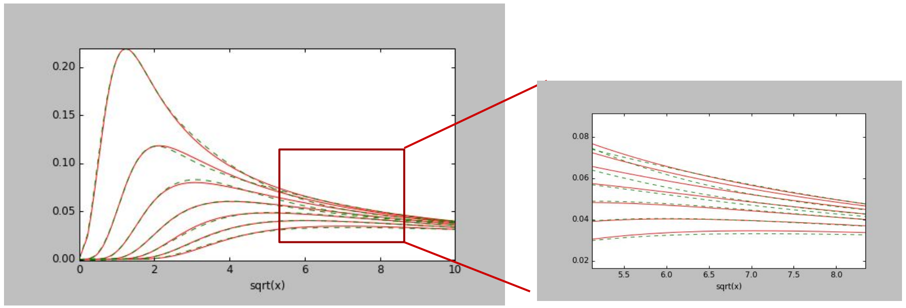
$$c_0^{(n)} + \sum_{i=1}^{i_{max}} \frac{c_i^{(n)}}{\lambda - d_i^{(n)}}$$

5 poles : $|err|_{max} < 0.45\%$ for $0 < k_{perp}\lambda < 10$

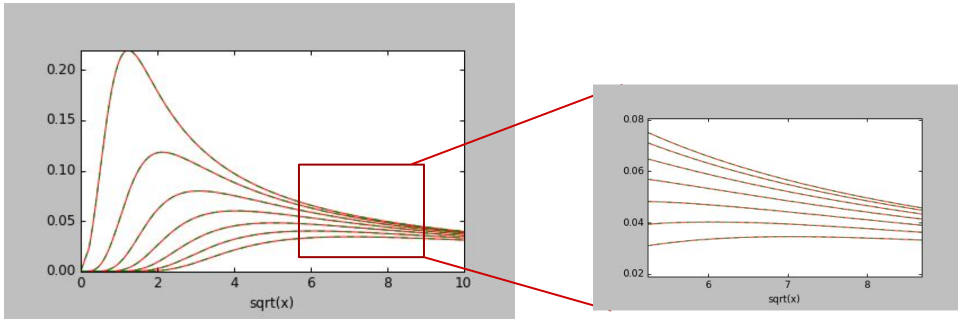
7 poles : $|err|_{max} < 0.036\%$ for $0 < k_{perp}\lambda < 10$

Fitting range

5 poles



7 poles



Self-adjointness can be maintained by symmetrizing the operator

Coupled linear system:

$$\begin{pmatrix} \nabla \times \nabla \times -\omega^2 \epsilon_{cold} & -i\omega \\ L_i & -(-\Delta_\perp - d_i) \end{pmatrix} \begin{pmatrix} E \\ J^{hot} \end{pmatrix} = \begin{pmatrix} i\omega J_{ant} \\ 0 \end{pmatrix}$$

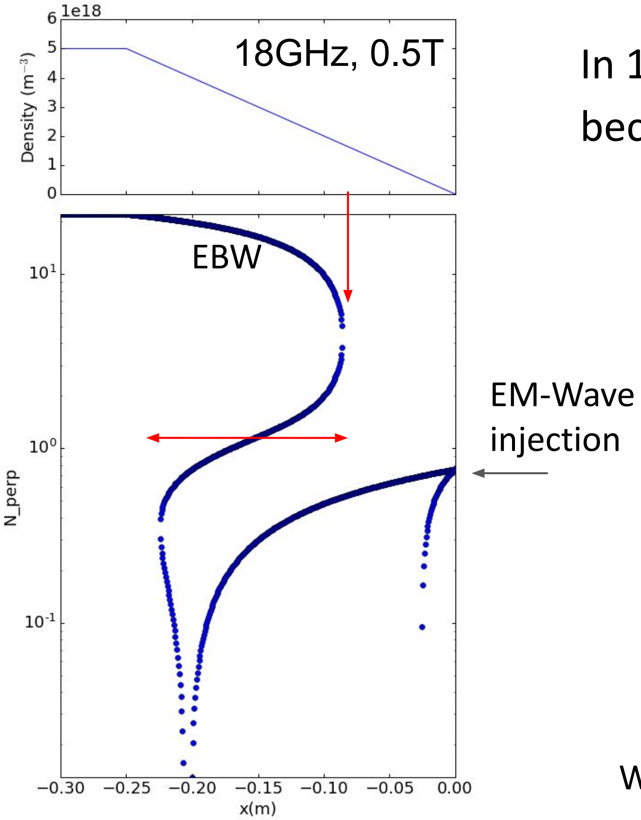
Spatial variation of plasma parameters breaks self-adjointness.

Symmetrizing the linear system:

$$\begin{pmatrix} \nabla \times \nabla \times -\omega^2 \epsilon_{cold} & -i\omega/2 & \overline{L_i}^T \\ L_i & -(-\Delta_\perp - d_i) & \\ i\omega/2 & & -(-\Delta_\perp - \overline{d_i}) \end{pmatrix} \begin{pmatrix} E \\ J^{hot(1)} \\ J^{hot(2)} \end{pmatrix} = \begin{pmatrix} i\omega J_{ant} \\ 0 \\ 0 \end{pmatrix}$$

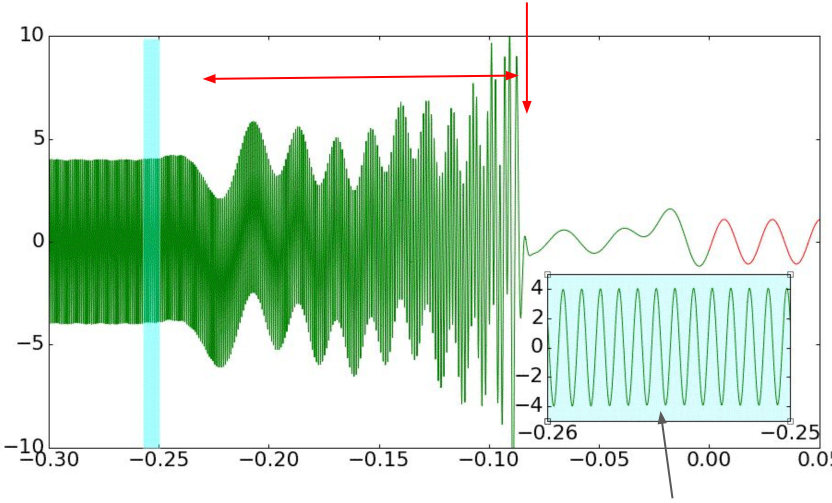
Linear system for electric field (after eliminating J) becomes Hermitian

Mode conversion (MC) of electron Bernstein waves (EBW) in 1D



In 1D, the operator assembly can be readily done with MFEM because

$$\Delta_\perp \Rightarrow \frac{\partial^2}{\partial x^2} - k_y^2$$



Wavelength ~ 0.75mm

Wave propagation features are captured well

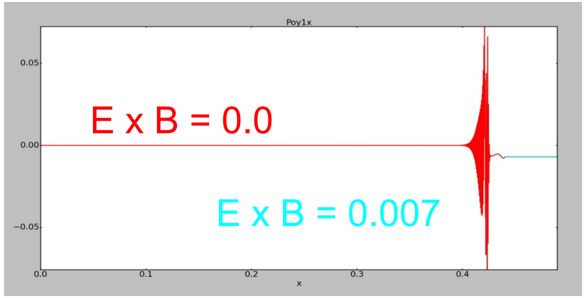
- Backward phase propagation of EBW.
- Wavelength matches with the dispersion.

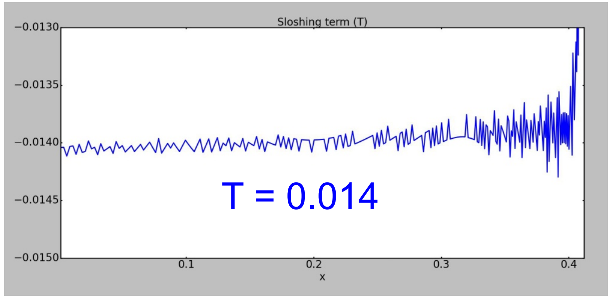
MC efficiency reproduces the one in literature and Poynting flux agrees with kinetic energy

Energy conservation :

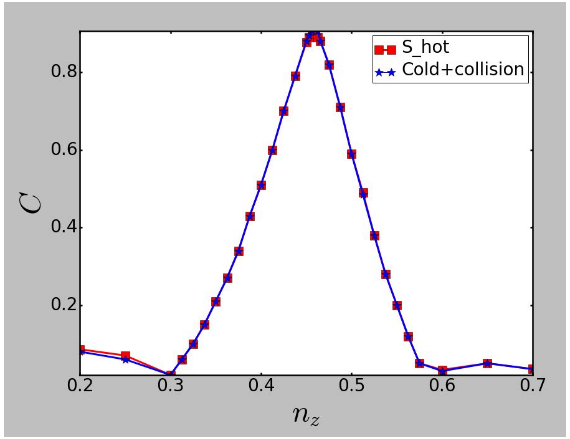
- Poynting flux $\mathbf{P} = \frac{1}{\mu_0}(\mathbf{E}^* \times \mathbf{B} + \mathbf{E} \times \mathbf{B}^*) = 2\frac{1}{\mu_0}\mathrm{Re}(\mathbf{E}^* \times \mathbf{B})$
- Kinetic energy (thermal motion of particles)

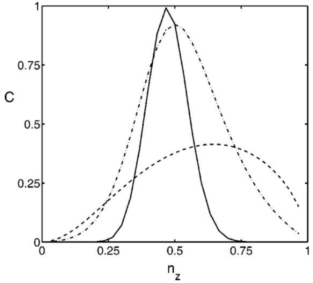
$$\mathbf{T} = -\omega\epsilon_0 \mathbf{E}^* \cdot \frac{\partial \epsilon_\mathrm{h}}{\partial \mathbf{k}} \cdot \mathbf{E}$$





$$\mathbf{P} = \mathbf{T}$$





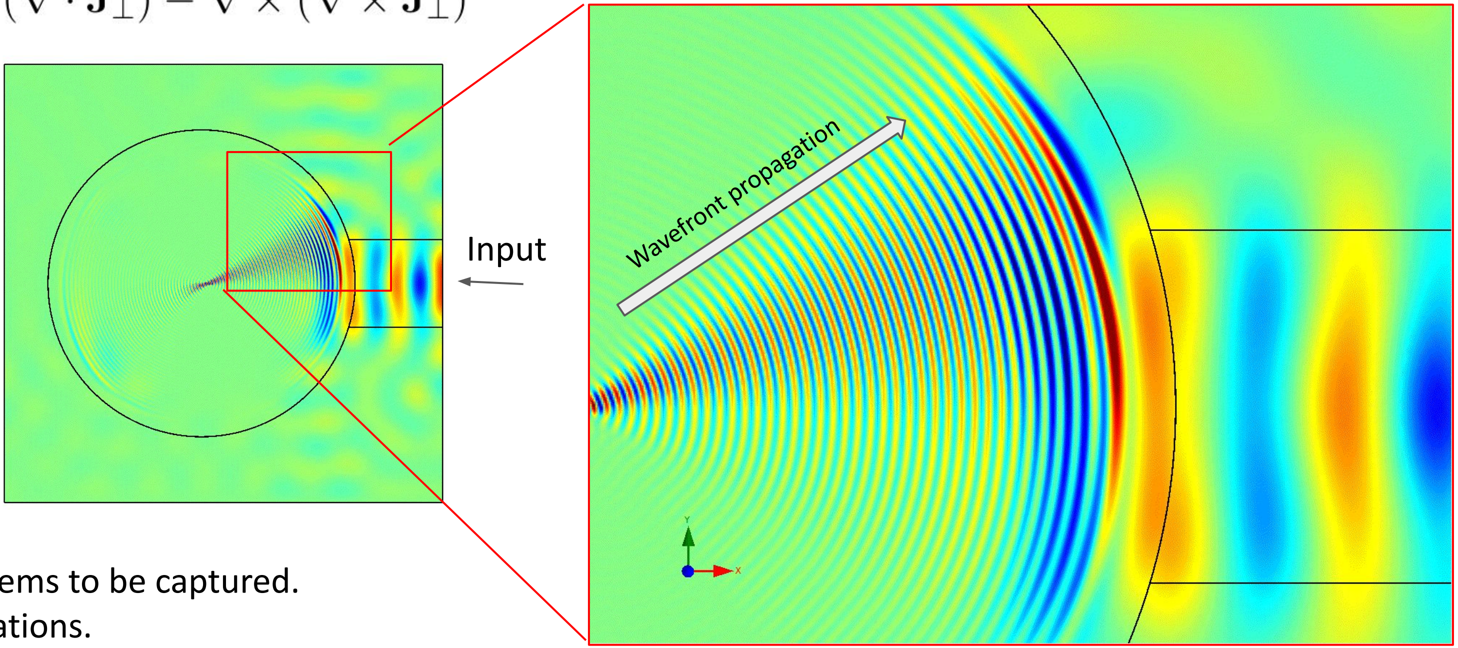
Ram & Schultz, Phys. Plasmas (2000)

Same operator can be used in 2D (preliminary)

In 2D (and 3D), the operator becomes the perpendicular vector Laplacian, which doesn't exist in MFEM. For now, we approximate it as

$$\Delta_\perp \Rightarrow \Delta$$

$$\Delta_\perp \mathbf{J} = \nabla(\nabla \cdot \mathbf{J}_\perp) - \nabla \times (\nabla \times \mathbf{J}_\perp)$$



Wave propagation seems to be captured.
Needs further verifications.

Summary

Constructing a differential operator to handle non-local plasma dielectric response in FEM:

- Based on rational approximation of dielectric tensor.
- Avoid convolution integral.
- Includes up-to 2nd order derivatives (no high order derivatives).

Test simulation of the mode conversion of electromagnetic wave to electron Bernstein wave shows:

- Wavelength agrees with the one expected from the dispersion relationship.
- 100% conversion of Poynting flux to kinetic flux.
- MC efficiency agrees with literature.
- 2D MC in progress.

<u>Questions and future work:</u>

- Does this approach scale to 3D?
- What is appropriate boundary condition if hot plasma touches the material?
- How about using more complicated dielectric response, such as relativistic and parallel dispersion?
- Can we derive directly the operator from kinetic theory?