High order positivity-preserving entropy stable discontinuous Galerkin discretizations

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Yimin has been the driving force behind the work in this talk.
High order finite element methods for hyperbolic PDEs

- Fluid dynamics applications: acoustics, vorticular flows, turbulence, shocks.
- Goal: high accuracy on unstructured meshes.
- Discontinuous Galerkin (DG) methods: geometric flexibility + high order.

Fluid dynamics applications: acoustics, vortical flows, turbulence, shocks.

Goal: high accuracy on unstructured meshes.

Discontinuous Galerkin (DG) methods: geometric flexibility + high order.
Why discontinuous Galerkin methods?

(a) High order FEM

(b) High order DG

High order DG mass matrices: easily invertible for explicit time-stepping.
Why high order accuracy?

Accurate resolution of propagating vortices and waves.
Why high order accuracy?

2nd, 4th, and 16th order Taylor-Green vortex. Vorticular structures and acoustic waves are both sensitive to numerical dissipation.

Results from Beck and Gassner (2013).
Why *not* high order DG methods?

(a) Exact solution

(b) 8th order DG

High order methods blow up for under-resolved solutions of nonlinear conservation laws (e.g., shocks and turbulence).
Why *not* high order DG methods?

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**Why not high order DG methods?**

- **(a)** Exact solution
- **(b)** 8th order DG

High order methods blow up for under-resolved solutions of nonlinear conservation laws (e.g., shocks and turbulence).
Why entropy stability for high order schemes?

- High order DG needs heuristic stabilization (e.g., artificial viscosity, filtering).
- Entropy stable schemes improve robustness without no added dissipation.
- Turns DG into a “good” high order method.

Finite volume methods: Tadmor, Chandrashekar, Ray, Svard, Fjordholm, Mishra, LeFloch, Rohde, ...
High order DGSEM: Fisher, Carpenter, Gassner, Winters, Kopriva, Persson, Pazner, ...
High order simplices: Chen and Shu, Crean, Hicken, Del Rey Fernandez, Zingg, ...
Examples of high order entropy stable DG simulations

All simulations are run without artificial viscosity, filtering, or slope limiting.

Parsani et al. (2021). *High-order accurate entropy-stable discontinuous collocated Galerkin...*
Talk outline

1. Entropy stable nodal DG methods
2. Entropy stable nodal DG with positivity-preserving limiting
3. From subcell limiting to a cell entropy inequality
Entropy stable nodal DG methods
Entropy stability for nonlinear problems

- Energy balance for **nonlinear** conservation laws (Burgers’, shallow water, compressible Euler + Navier-Stokes).

\[
\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0.
\]

- Continuous entropy inequality: convex entropy function \(S(u)\), entropy potential \(\psi(u)\), entropy variables \(v(u)\)

\[
\int_\Omega v^T \left( \frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} \right) = 0,
\]

\[
\implies \int_\Omega \frac{\partial S(u)}{\partial t} + (v^T f(u) - \psi(u)) \bigg|_{-1}^1 \leq 0.
\]
entropy conservative finite volume methods

- Finite volume scheme:

\[
\frac{d\mathbf{u}_i}{dt} + \frac{f(u_{i+1}, \mathbf{u}_i) - f(u_i, \mathbf{u}_{i-1})}{h} = 0.
\]

- Take \( f = f_{EC} \) to be an entropy conservative numerical flux

\[
f_{EC}(\mathbf{u}, \mathbf{u}) = f(\mathbf{u}), \quad \text{(consistency)}
\]

\[
f_{EC}(\mathbf{u}, \mathbf{v}) = f_{EC}(\mathbf{v}, \mathbf{u}), \quad \text{(symmetry)}
\]

\[
(\mathbf{v}_L - \mathbf{v}_R)^T f_{EC}(\mathbf{u}_L, \mathbf{u}_R) = \psi_L - \psi_R, \quad \text{(conservation)}.
\]

- Can show numerical scheme conserves entropy

\[
\int_{\Omega} \frac{\partial S(\mathbf{u})}{\partial t} \approx \sum_i h \frac{dS(\mathbf{u}_i)}{dt} = 0.
\]

Example of EC fluxes (compressible Euler equations)

- Define average $\{\{u\}\} = \frac{1}{2}(u_L + u_R)$. In one dimension:

  $$f^1_S(u_L, u_R) = \{\{\rho\}\} \log \{\{u\}\}$$
  $$f^2_S(u_L, u_R) = \{\{u\}\} f^1_S + p_{avg}$$
  $$f^3_S(u_L, u_R) = (E_{avg} + p_{avg}) \{\{u\}\},$$

  $$p_{avg} = \frac{\{\{\rho\}\}}{2 \{\{\beta\}\}} , \quad E_{avg} = \frac{\{\{\rho\}\} \log (\gamma - 1)}{2 \{\{\beta\}\} \log (\gamma - 1)} + \frac{1}{2} u_L u_R.$$

- Non-standard logarithmic mean, “inverse temperature” $\beta$

  $$\{\{u\}\} \log = \frac{u_L - u_R}{\log u_L - \log u_R}, \quad \beta = \frac{\rho}{2p}.$$

Chandreshkar (2013), *Kinetic energy preserving and entropy stable finite volume schemes for the compressible Euler and Navier-Stokes equations.*
Matrix reformulation using Hadamard products

Hadamard product of two matrices $A \circ B$

$$
\begin{bmatrix}
A_{11} & \cdots & A_{1n} \\
\vdots & \ddots & \vdots \\
A_{n1} & \cdots & A_{nn}
\end{bmatrix} \circ
\begin{bmatrix}
B_{11} & \cdots & B_{1n} \\
\vdots & \ddots & \vdots \\
B_{n1} & \cdots & B_{nn}
\end{bmatrix} =
\begin{bmatrix}
A_{11}B_{11} & \cdots & A_{1n}B_{1n} \\
\vdots & \ddots & \vdots \\
A_{n1}B_{n1} & \cdots & A_{nn}B_{nn}
\end{bmatrix}.
$$

Rewrite an $N$-point (periodic) finite volume scheme as

$$
\frac{d}{dt} \begin{bmatrix}
u_1 \\
u_2 \\
\vdots \\
u_N
\end{bmatrix} + \frac{1}{h} \begin{bmatrix}
f_{EC}(u_1, u_2) - f_{EC}(u_N, u_1) \\
f_{EC}(u_2, u_3) - f_{EC}(u_1, u_2) \\
\vdots \\
f_{EC}(u_N, u_1) - f_{EC}(u_{N-1}, u_N)
\end{bmatrix} = 0.
$$
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Rewrite an $N$-point (periodic) finite volume scheme as

$$
\begin{aligned}
\frac{h}{d} \frac{d}{dt} 
\begin{bmatrix}
\mathbf{u}_1 \\
\mathbf{u}_2 \\
\vdots \\
\mathbf{u}_N
\end{bmatrix}
+ 
\begin{bmatrix}
\mathbf{F}_{1,2} - \mathbf{F}_{1,N} \\
\mathbf{F}_{2,3} - \mathbf{F}_{2,1} \\
\vdots \\
\mathbf{F}_{N,1} - \mathbf{F}_{N,N-1}
\end{bmatrix}
= 0,
\end{aligned}
$$

where $\mathbf{F}_{ij} = f_{EC}(\mathbf{u}_i, \mathbf{u}_j)$. 

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\vdots \\
F_{N,1} - F_{N,N-1}
\end{bmatrix}
= 
\begin{bmatrix}
0 & 1 & \ldots & -1 \\
-1 & 0 & 1 & \vdots \\
\vdots & \ddots & \ddots & \ddots \\
1 & \ldots & \ldots & 1 \\
1 & -1 & 0
\end{bmatrix}
\circ
\begin{bmatrix}
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\begin{bmatrix}
F_{1,2} - F_{1,N} \\
F_{2,3} - F_{2,1} \\
\vdots \\
F_{N,1} - F_{N,N-1}
\end{bmatrix} = 2(Q \circ F)1.
$$
Interpretation using finite difference matrices

Let \( M = hI \). Can reformulate entropy conservative finite volume as

\[
M \frac{du}{dt} + 2 (Q \circ F) 1 = 0, \quad Q = \frac{1}{2} \begin{bmatrix}
0 & 1 & -1 \\
-1 & 0 & 1 \\
1 & -1 & 0
\end{bmatrix}
\]

Note: \( M^{-1}Q \) is a 2nd order (periodic) differentiation matrix.

Key result: generalizable beyond finite volumes

Entropy conservation for any \( Q = -Q^T \) and \( Q1 = 0 \)!
Interpretation using finite difference matrices

Let $M = hI$. Can reformulate entropy conservative finite volume as

$$M \frac{du}{dt} + 2 (Q \circ F) 1 = 0,$$

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Key result: generalizable beyond finite volumes

Entropy conservation for any $Q = -Q^T$ and $Q1 = 0$!
A brief intro to nodal discontinuous Galerkin methods

- Multiply by nodal (Lagrange) basis $\ell_i(x)$ and integrate

$$\int_{D^k} \left( \frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} \right) \ell_i + \int_{\partial D^k} (f^*(u^+, u^-) - f(u^-)) n \ell_i = 0$$

- The numerical flux $f^*(u^+, u^-) \approx f(u)$ enforces boundary conditions and weak continuity across interfaces.

- Nodal (collocation) DG methods: use Gauss-Lobatto quadrature nodes for both interpolation and integration.
Matrix formulation of nodal DG methods

- Map integrals to the reference interval $\hat{D} = [-1, 1]$

$$\int_{\hat{D}} \left( \frac{h}{2} \frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} \right) \ell_i + \int_{\partial \hat{D}} (f^*(u^+, u^-) - f(u^-)) n \ell_i = 0$$

- Use $u(x, t) = \sum_j u_j(t) \ell_j(x)$ and $\int_{-1}^{1} \frac{\partial f(u)}{\partial x} \ell_i \approx Qf(u)$

$$M \frac{du}{dt} + Qf(u) + E^T B \left( f^*(u^+, u^-) - f(u^-) \right) = 0.$$  

where $M = \frac{h}{2} \text{diag}(w_1, \ldots, w_{N+1})$, and $Q, B, E$ are differentiation and boundary matrices

$$Q_{ij} = \int_{-1}^{1} \frac{\partial \ell_j}{\partial x} \ell_i, \quad B = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad E = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix}. $$
A “flux differencing” formulation

• Idea: reformulate the DG flux derivative term

\[
\int_{-1}^{1} \frac{\partial f(u)}{\partial x} \ell_i \approx Q f(u).
\]

• Note that \( Q1 = 0 \), so \( \sum_j Q_{ij} = 0 \). Thus,

\[
(Q f(u))_i = \sum_j Q_{ij} (f(u_j) + f(u_i)) = 2 \sum_j Q_{ij} \frac{f(u_j) + f(u_i)}{2}
\]

• We replace the central flux with an entropy conservative flux

\[
2 \sum_j Q_{ij} f_{EC}(u_i, u_j) = (2 (Q \circ F) 1)_i, \quad F_{ij} = f_{EC}(u_i, u_j).
\]
Extension to multiple elements

• An entropy stable nodal DG formulation can be written as:

$$M \frac{du}{dt} + Qf(u) + E^T B \left( f^* (u^+, u^-) - f(u^-) \right) = 0.$$

• If $Q$ satisfies the summation-by-parts (SBP) property

$$Q + Q^T = E^T B E$$

and if $f^* (u^+, u^-)$ is entropy stable (e.g., local Lax-Friedrichs flux), a cell entropy inequality holds:

$$\int_{D^k} \frac{\partial S(u)}{\partial t} + \int_{\partial D^k} (v^T f^* (u^+, u^-) - \psi (u)) n \leq 0.$$

An entropy stable nodal DG formulation can be written as:

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M \frac{du}{dt} + 2 (Q \circ F) 1 + E^T B \left( f^*(u^+, u^-) - f(u^-) \right) = 0.
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Extension to multiple elements

- An entropy stable nodal DG formulation can be written as:

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\]

Some of our recent work on entropy stable methods

- More general entropy stable “modal” DG formulations.
- Network domains, reduced order modeling
- Non-conforming meshes (Mario Bencomo, DCDR Fernandez)
- Positivity preserving entropy stable schemes for compressible Navier-Stokes (Yimin Lin, T. Warburton, I. Tomas)
- Efficient implementations (with the developers of Trixi.jl)

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Lin, Chan, Tomas (2023). *A positivity preserving strategy for entropy stable discontinuous Galerkin discretizations of the compressible Euler and Navier-Stokes equations.*
Entropy stable nodal DG with
positivity-preserving limiting
Entropy stable schemes require positivity of density, pressure (numerical fluxes depend on logarithm of density, temperature).

Interpretation of Lobatto nodes as a sub-cell finite volume grid.

- Hard to enforce both high order accuracy and positivity.
- Strategy: blend high order method with a first order positive method to retain conservation and subcell resolution.

The low order method: a global matrix formulation

Start with a naive *global* matrix formulation using forward Euler (can extend to higher order in time using SSP-RK).

An example nodal DG discretization.

\[ m_i \frac{u_i^{k+1} - u_i}{\Delta t} + \sum_{j \in N(i)} Q_{ij} f(u_j) = 0. \]

Equivalent to a central DG scheme.

Can show that \( \sum_j Q_{ij} = 0 \) and (for periodic domains) \( Q_{ij} = -Q_{ji} \).
Enforcing positivity: a first order positive subcell scheme

Add dissipation to our global matrix formulation with forward Euler (or SSP-RK). Let $d_{ij} = d_{ji} > 0$ for $i \neq j$, $\sum_j d_{ij} = 0$.

$$
m_i \frac{u_{i}^{k+1} - u_i}{\Delta t} + \sum_{j \in N(i)} Q_{ij} f(u_j) - d_{ij} (u_j - u_i) = 0. \quad (1)
$$

(Equivalent to DG with LxF interface fluxes + volume dissipation)

Use properties of $Q$ to rewrite (1) in terms of “bar states”

$$
\bar{u}_{ij} = \frac{1}{2} (u_i + u_j) - \frac{Q_{ij}}{d_{ij}} (f_j - f_i), \quad \text{where } f_j = f(u_j),
$$

$$
m_i \frac{u_{i}^{k+1}}{\Delta t} = \left( \frac{m_i}{\Delta t} - \sum_{j \neq i} 2d_{ij} \right) u_i + \sum_{j \neq i} \frac{2\Delta t d_{ij}}{m_i} \bar{u}_{ij}.
$$

Guermond, Popov, and Tomas (2019). Invariant domain preserving discretization-independent schemes and convex limiting for hyperbolic systems.
Enforcing positivity: a first order positive subcell scheme

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$$m_i \frac{u_i^{k+1} - u_i}{\Delta t} + \sum_{j \in N(i)} Q_{ij} f(u_j) - \frac{d_{ij}}{m_i} (u_j - u_i) = 0.$$ (1)

(Equivalent to DG with LxF interface fluxes + volume dissipation)

Use properties of $Q$ to rewrite (1) in terms of “bar states”

$$\bar{u}_{ij} = \frac{1}{2} (u_i + u_j) - \frac{Q_{ij}}{d_{ij}} (f_j - f_i), \quad \text{where } f_j = f(u_j),$$

$$\frac{m_i}{\Delta t} u_i^{k+1} = \left( \frac{m_i}{\Delta t} - \sum_{j \neq i} 2d_{ij} \right) u_i + \sum_{j \neq i} \frac{2\Delta t d_{ij}}{m_i} \bar{u}_{ij}. $$

Guermond, Popov, and Tomas (2019). Invariant domain preserving discretization-independent schemes and convex limiting for hyperbolic systems.
Provable positivity under a CFL condition

\[
\frac{m_i}{\Delta t} u_i^{k+1} = \left( \frac{m_i}{\Delta t} - \sum_{j \neq i} 2d_{ij} \right) u_i + \sum_{j \neq i} \frac{2\Delta t d_{ij}}{m_i} \bar{u}_{ij}.
\]

- Bar states \(\bar{u}_{ij}\) resemble a Lax-Friedrichs intermediate state, and \textbf{preserve positivity} if \(d_{ij}\) is sufficiently large

\[
\bar{u}_{ij} = \frac{1}{2} (u_i + u_j) - \frac{Q_{ij}}{d_{ij}} (f_j - f_i), \quad d_{ij} \geq \lambda_{\max} (u_i, u_j, Q_{ij}).
\]

- \(u_i^{k+1}\) is positive (a convex combination of \(u_i\) and \(\bar{u}_{ij}\)) if

\[
\Delta t \leq \min_i \frac{m_i}{2 \sum_{i \neq j} d_{ij}}.
\]

Guermond and Popov (2016). \textit{Invariant domains and first-order continuous finite element approximation for hyperbolic systems.}
Our work: extension to compressible Navier-Stokes

• Entropy stable discretization of viscous terms $\sigma$, which include the stress $\tau +$ heat conduction $q$.

$$M\frac{du}{dt} + \sum_j Q_{ij}(f_j - \sigma_j) - d_{ij}(u_j - u_i) = 0.$$  

• Reformulate scheme in terms of viscous bar states:

$$\bar{u}_{ij} = \frac{1}{2}(u_i + u_j) - \frac{Q_{ij}}{d_{ij}}((f_j - \sigma_j) - (f_i - \sigma_i))$$

• Positivity of $\rho, p$ under a (viscous) CFL condition with

$$d_{ij} = \max(\beta(u_i), \beta(u_j), \lambda_{\max}(u_i, u_j, Q_{ij}), \lambda_{\max}(u_j, u_i, Q_{ji}))$$

$$\beta(u) > |v \cdot n| + \frac{1}{2\rho^2 e} \left(\sqrt{\rho^2(q \cdot n)^2 + 2\rho^2 e \|\tau \cdot n - pn\|}\right) + \rho |q \cdot n|$$

Sparsification of low order discretization matrices

Low order advection solutions with (solid) and without (dashed) sparsification (from Pazner 2021).

- **Algebraic** artificial dissipation depends on discretization matrices $\Rightarrow$ dense operators produce too much diffusion!
- Solution: use *sparse* SBP operators in the low order method.

Sparsification of low order discretization matrices

\[
Q = \frac{1}{2} \begin{bmatrix}
-1 & 1 \\
-1 & 0 & 1 \\
\vdots & \vdots & \vdots \\
-1 & 0 & 1 \\
-1 & 1
\end{bmatrix}
\]

\[Q1 = 0, \quad Q + Q^T = E^T B E\]

summation-by-parts property

- **Algebraic** artificial dissipation depends on discretization matrices \(\implies\) dense operators produce too much diffusion!
- Solution: use *sparse* SBP operators in the low order method.

Constructing sparse low order simplicial SBP operators

- Want to preserve conservation $Q^{\text{low}} \mathbf{1} = \mathbf{0}$ and SBP property
  \[ Q^{\text{low}} + \left( Q^{\text{low}} \right)^T = B. \]

- For neighboring $i$ and $j$, assume
  \[ \left( Q^{\text{low}} - \left( Q^{\text{low}} \right)^T \right)_{ij} = \psi_j - \psi_i. \]

- Enforcing $Q^{\text{low}} \mathbf{1} = \mathbf{0}$ equivalent to
  \[ \sum_j \psi_j - \psi_i = \left( -\frac{1}{2} B \mathbf{1} \right)_i, \]
  \[ \text{s.t. } \psi^T \mathbf{1} = 0. \]

Quadrature nodes from Chen, Shu (2017) for a degree $N = 3$ SBP operator. The sparse low order operator $Q^{\text{low}}$ uses the same nodes and weights.
• Blend high and low order solutions over each element to retain accuracy where possible while ensuring positivity.

\[ u^{k+1} = (1 - \ell)u^{k+1,\text{low}} + \ell u^{k+1,\text{high}} \]

• Impose minimal local bounds based on low order solution with relaxation factor \( \alpha \)

\[ \rho \geq \alpha \rho^{\text{low}}, \quad p \geq \alpha p^{\text{low}}, \quad \alpha \in [0, 1]. \]

• Local entropy inequality: preserved for element-wise blending.

• Local conservation: preserved if high and low order schemes use the same interface flux.
Convergence tests: LeBlanc and viscous shock tube

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<td>L^1 error</td>
<td>Rate</td>
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<td>0.94</td>
<td>5.889 × 10^{-4}</td>
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(a) Leblanc shock tube, relaxation factor $\alpha = 0.5$

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(b) 1D viscous shock, Re = 1000, relaxation factor $\alpha = 0.5$

Viscous shock is run at Mach 20 to generate positivity violations.
Isentropic vortex with small minimum density

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<td>$L^2$ error</td>
<td>Rate</td>
<td>$L^2$ error</td>
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(c) Quadrilateral meshes, relaxation factor $\alpha = 0.5$

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<td>Rate</td>
<td>$L^2$ error</td>
</tr>
<tr>
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<td>$2.606 \times 10^{-3}$</td>
<td>3.50</td>
<td>$4.258 \times 10^{-4}$</td>
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(d) Triangular meshes, relaxation factor $\alpha = 0.5$

Challenging vortex parameters: $\rho_{\text{min}} = 2.145 \times 10^{-3}$
Compressible Euler: double Mach reflection

(a) Subcell positivity-preserving entropy stable nodal DG, $\alpha = 0.5, T = 0.2$

(b) Subcell invariant domain preserving nodal DG (Pazner 2021), $T = 0.275$

Density for $N = 3$ entropy stable DG ($250 \times 875$ elements) and a reference solution ($600 \times 2400$ elements). Note: positivity is sensitive to the wall boundary treatment!

Compressible Euler: Sedov blast wave

(a) $\alpha = 0.1$

(b) $\alpha = 0.5$

(c) $\alpha = 0.1 +$ shock capturing

Quadrilateral meshes with $100^2$ degree $N = 3$ elements.
Compressible Euler: Sedov blast wave

(a) $\alpha = 0.1$

(b) $\alpha = 0.5$

(c) $\alpha = 0.1 + \text{shock capturing}$

Triangular meshes with $100^2$ degree $N = 3$ elements.

Guermond, Kronbichler, Maier, Popov, Tomas (2022). *On the implementation of a robust and efficient finite element-based parallel solver for the compressible Navier-Stokes equations.*
Many phenomena are sensitive to “shock capturing”

(a) Degree $N = 3$, $600 \times 300$ grid
(b) Degree $N = 2$, $400 \times 200$ grid
(c) With entropy stable shock capturing ($N = 2$, $400 \times 200$)
(d) Dzanic and Witherden ($N = 4$, $800 \times 400$ grid)


Sensitivity to shock capturing at higher Reynolds numbers

Degree $N = 2$, $200 \times 100$ mesh with positivity parameter $\alpha = 0.1$.

Same setup with positivity parameter $\alpha = 0.5$.

Hennemann et al. (2021). “A provably ES subcell shock capturing approach for high order split form DG for the compressible Euler equations”.
Degree $N = 2$, $200 \times 100$ mesh with positivity parameter $\alpha = 0.1$.

Positivity parameter $\alpha = 0.1$ with Hennemann (2021) shock capturing.

Hennemann et al. (2021). “A provably ES subcell shock capturing approach for high order split form DG for the compressible Euler equations”.
From subcell limiting to a cell entropy inequality
Subcell *resolution* vs subcell *blending*

- The low order method has subcell *resolution*, but element-wise constant high/low order blending.
- It is possible to perform blending at a *subcell* level; however, subcell blending does not necessarily preserve entropy stability!
- Not all entropy inequalities preserve high order accuracy ...
Start with a (possibly non-ES) semi-discrete nodal DG formulation:

\[ m_i \frac{du_i}{dt} + r_i + (\delta_{i,N+1} f_R^* - \delta_{i,1} f_L^*) = 0, \quad \sum r_j = 0 \]

This is algebraically equivalent to the following scheme

\[ m_i \frac{du_i}{dt} + \bar{f}_{i+1} - \bar{f}_i = 0, \]

where \( \bar{f}_i \) are “reconstructed” finite volume-type fluxes

\[ \bar{f}_1 = f_L^* \]

\[ \bar{f}_{1+i} = \sum_{j=1}^{i} r_j, \quad i = 1, \ldots, N + 1 \]

\[ \bar{f}_{N+2} = f_R^*. \]
Blend high and low order schemes with subcell fluxes.

\[
\begin{align*}
\mathbf{m}_i \frac{d\mathbf{u}_i}{dt} &+ \left[ l_{i+1} \overline{f}^H_{i+1} + (1 - l_{i+1}) \overline{f}^L_{i+1} \right] \\
- \left[ l_i \overline{f}^H_i + (1 - l_i) \overline{f}^L_i \right] &= 0,
\end{align*}
\]

- \( l_i = 1 \implies \) recovers high order nodal DG.
- \( l_i = 0 \implies \) recovers low order invariant domain scheme.
- Pick largest \( l_i \) to satisfy positivity constraints while retaining as much of the high order method as possible.

Enforcing entropy stability as a pointwise constraint

- Option 1: minimum entropy principle (fully discrete).

\[ s_i \geq \min_{j \in N(i)} s_n^j. \]

- Option 2: semi-discrete condition on subcell fluxes

\[ (v_i - v_{i-1})^T \bar{f}_i \leq \psi(u_i) - \psi(u_{i-1}) \]

<table>
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<tr>
<th>K</th>
<th>(N = 2) (L^2) error</th>
<th>Rate</th>
<th>(N = 3) (L^2) error</th>
<th>Rate</th>
<th>(N = 4) (L^2) error</th>
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<td>2.157 \times 10^{-2}</td>
<td>0.95</td>
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</tbody>
</table>

Neither “pointwise” approach retains high order accuracy.

---

Tadmor, Eitan. “A minimum entropy principle in the gas dynamics equations.”

Kuzmin, Dmitri et al., “Limiter-based entropy stabilization of semi-discrete and fully discrete schemes for nonlinear hyperbolic problems.”
Enforcing a cell entropy inequality

Choose $l_1, \ldots, l_{N+2}$ over each element to enforce

$$v^TM \frac{du}{dt} \leq (\psi(u_{N+1}) - v^T_{N+1}f_R^*) - (\psi(u_1) - v^T_1f_L^*)$$

$$\Rightarrow \sum_{i=1}^{N+1} v_i^T (\bar{f}_{i+1}(l_{i+1}) - \bar{f}_i(l_i)) \leq \psi(u_{N+1}) - \psi(u_1),$$

where $\bar{f}_i(l_i) = l_i \bar{f}_i^H + (1 - l_i) \bar{f}_i^L$.

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Enforcing a cell entropy inequality recovers high order accuracy.
Formulation as an optimization problem

Maximizing limiting factors \( l_i \) while enforcing cell entropy inequality

\[
\sum_{i=1}^{N+1} v_i^T (\bar{f}_{i+1}(l_{i+1}) - \bar{f}_i(l_i)) \leq \psi(u_{N+1}) - \psi(u_1)
\]

can be posed as a continuous knapsack problem, which admits a fast and explicit \( O(n \log(n)) \) solution algorithm.

\[
\begin{align*}
\max_{l_i} & \quad \sum_{i=1}^{N} l_i \\
\text{s.t.} & \quad \sum_{i=1}^{N} a_i l_i \leq b \\
& \quad 0 \leq l_i \leq l_i^C
\end{align*}
\]

where \( l_i^C \leq 1 \) is a limiting factor upper bound to ensure positivity.
Modified Sod shock tube

Naive (DGSEM) and entropy stable (ESDG) DG discretizations with different limiting strategies for enforcing an entropy principle.

An entropy glitch appears without some form of entropy inequality (e.g., minimum entropy principle or cell entropy inequality).
2D KPP problem

(a) Shock capturing only
(b) Shock capturing and cell entropy inequality

Hennemann (2021) shock capturing with and without cell entropy inequality (degree $N = 3, 128 \times 128$ elements).

Kelvin Helmholtz instability

Long-time KHI simulation with positivity and cell entropy inequality enforced.
Compressible Euler - Mach 2000 Astrophysical jet

(a) Subcell limited DGSEM + cell entropy inequality
(b) Subcell limited ESDG + cell entropy inequality
(c) ESDG with element-wise positivity limiting

Density for degree $N = 3$, $150 \times 150$ elements. We enforce the relaxed positivity bounds $\rho > 0.5\rho^L$, $\rho e > 0.5 (\rho e)^L$. 
Conclusions and acknowledgements

- Positivity preserving limiters enable robust entropy stable nodal DG simulations of compressible flow.
- We can enforce a (high order?) cell entropy inequality for standard nodal DG using subcell limiting.

This work is supported by DMS-1943186 and DMS-2231482.

Thank you! Questions?


Lin, Chan, Tomas (2022). *A positivity preserving strategy for entropy stable discontinuous Galerkin discretizations of the compressible Euler and Navier-Stokes equations.*
Additional slides
Subcell limiting for compressible Navier-Stokes

(a) $N = 3$, $600 \times 300$ grid. 
ESDG + element-wise limiting.

(b) $N = 3$, $300 \times 150$ grid. 
ESDG + element-wise limiting.

(c) $N = 3$, $300 \times 150$ grid. 
DGSEM + subcell limiting

(d) $N = 3$, $300 \times 150$ grid. 
ESDG + subcell limiting

Daru-Tenaud simulations with positivity and cell entropy inequality.