



Conforming NURBS meshes with non-conforming patch topology

FEM@LLNL Seminar
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Outline

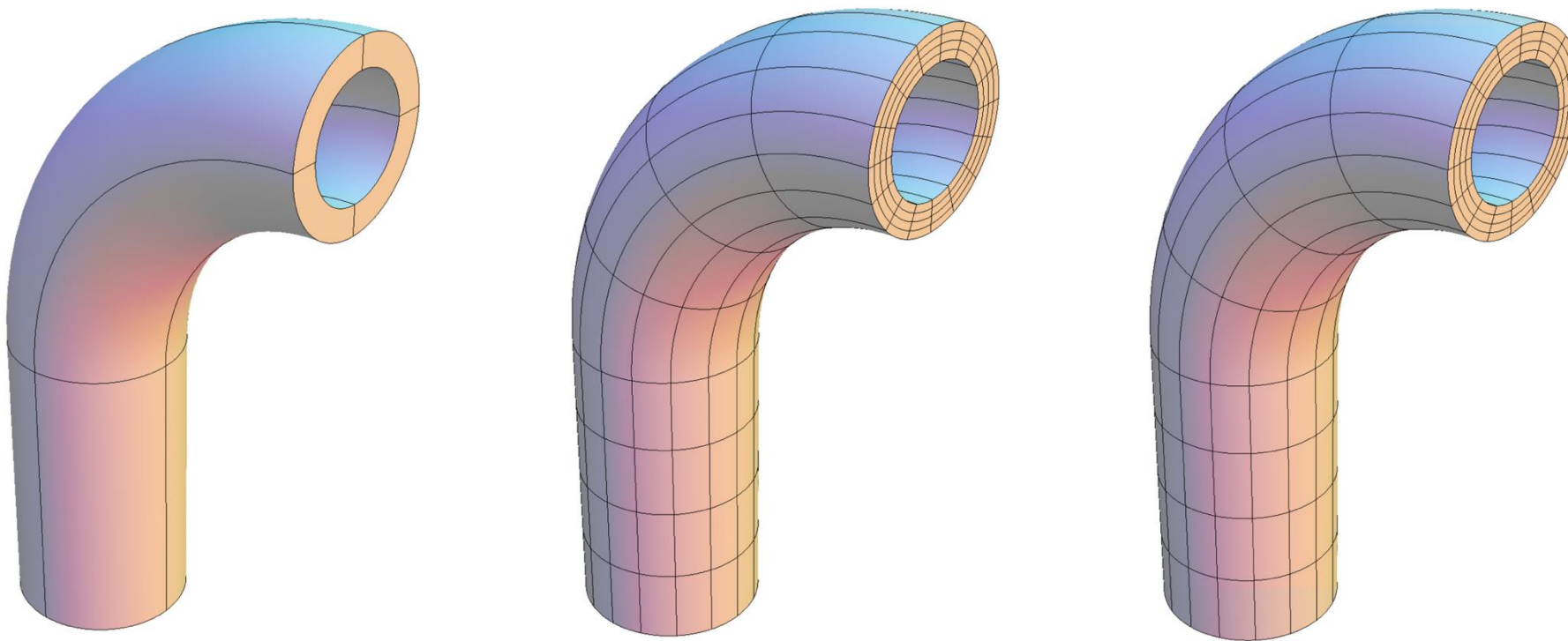
- NURBS meshes
- Non-conforming patches
- Spacing formulas
- Refinement by local factors
- NC-patch mesh generation
- Examples



NURBS meshes

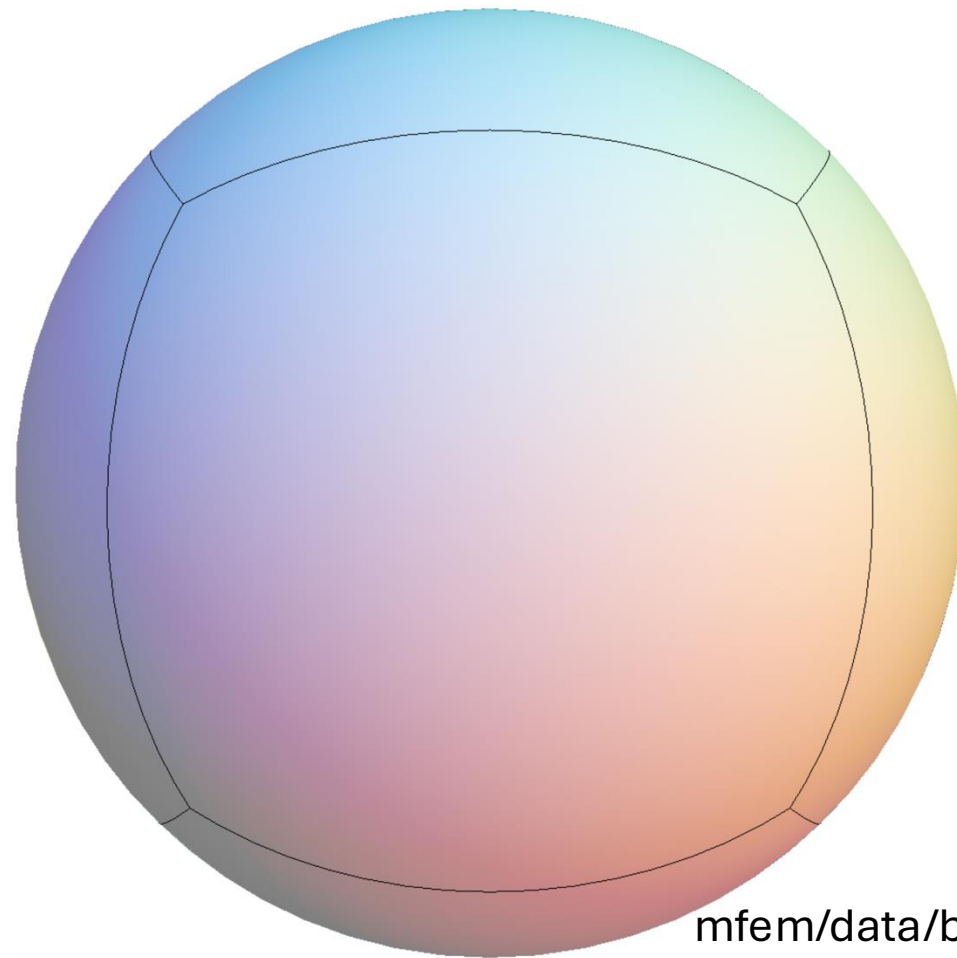
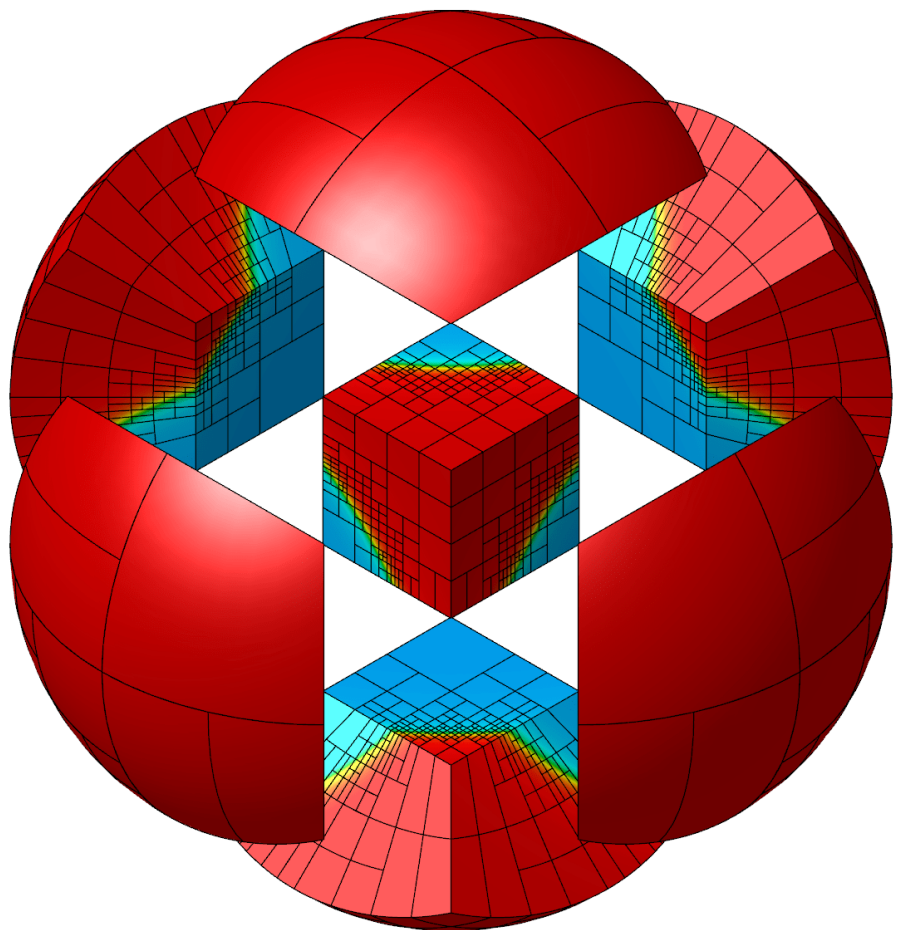
NURBS enable accurate coarse meshing

- Coarse NURBS -> refined NURBS -> high-order FEM mesh



Fine mesh may not be NURBS.

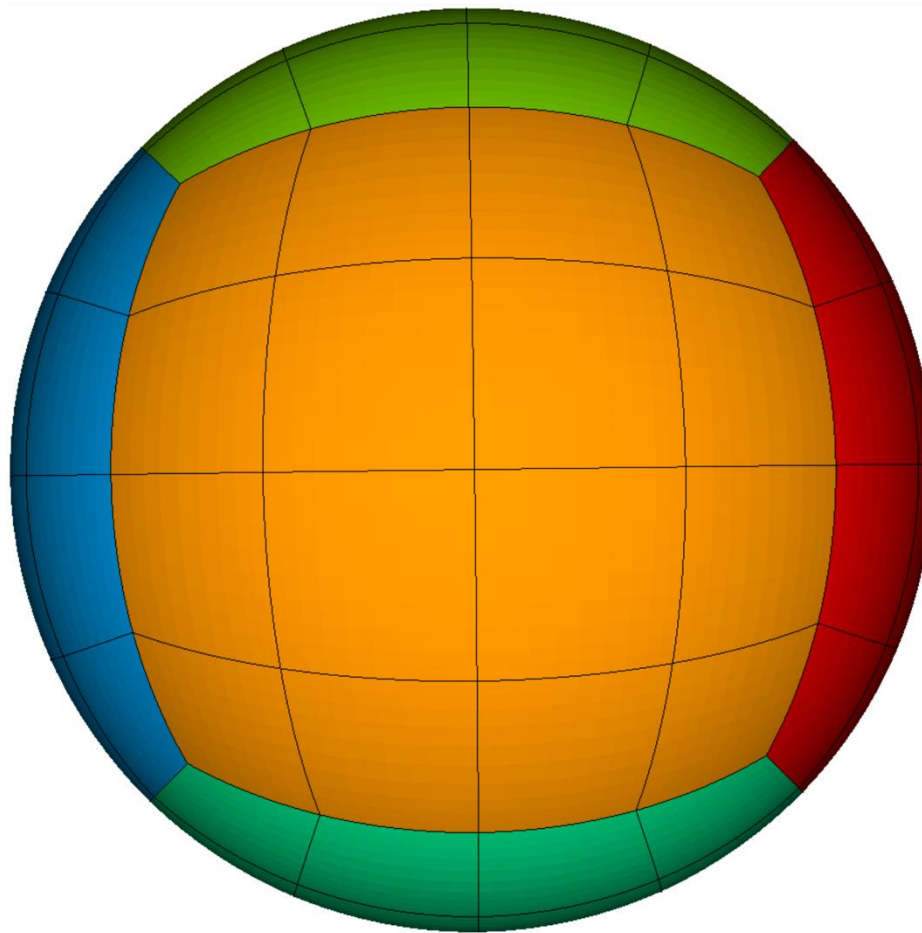
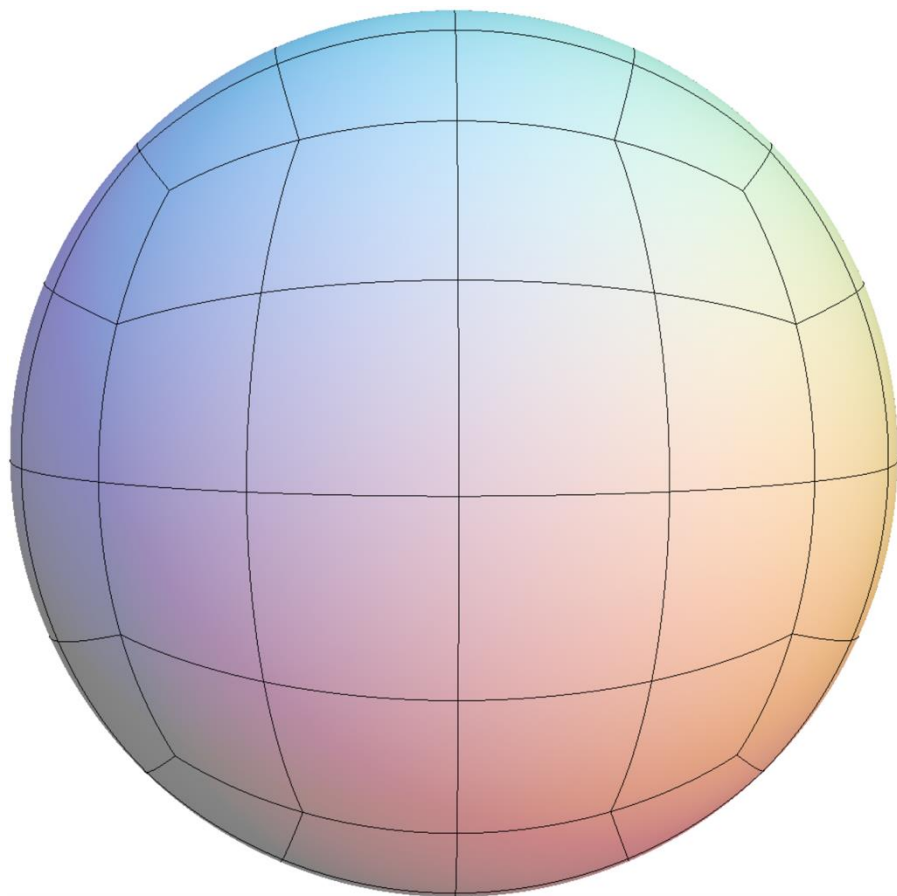
NURBS meshes consist of tensor-product patches



mfem/data/ball-nurbs.mesh

MFEM logo is an AMR mesh converted from a coarse NURBS mesh.

Exact conic sections with NURBS



Circles, ellipses, cylinders, spheres, cones, etc.

B-spline basis of degree p

- Knots define reference-space intervals and basis functions.

$$U = (a, \dots, a, u_{p+1}, \dots, u_{m-p-1}, b, \dots, b)$$

- Basis recurrence:

$$N_{i,0}(u) = \begin{cases} 1 & \text{if } u_i \leq u < u_{i+1} \\ 0 & \text{otherwise} \end{cases}$$

$$N_{i,p}(u) = \frac{u - u_i}{u_{i+p} - u_i} N_{i,p-1}(u) + \frac{u_{i+p+1} - u}{u_{i+p+1} - u_{i+1}} N_{i+1,p-1}(u)$$

B-spline basis with 4 elements, degree 3

- Interior knot multiplicity 1.

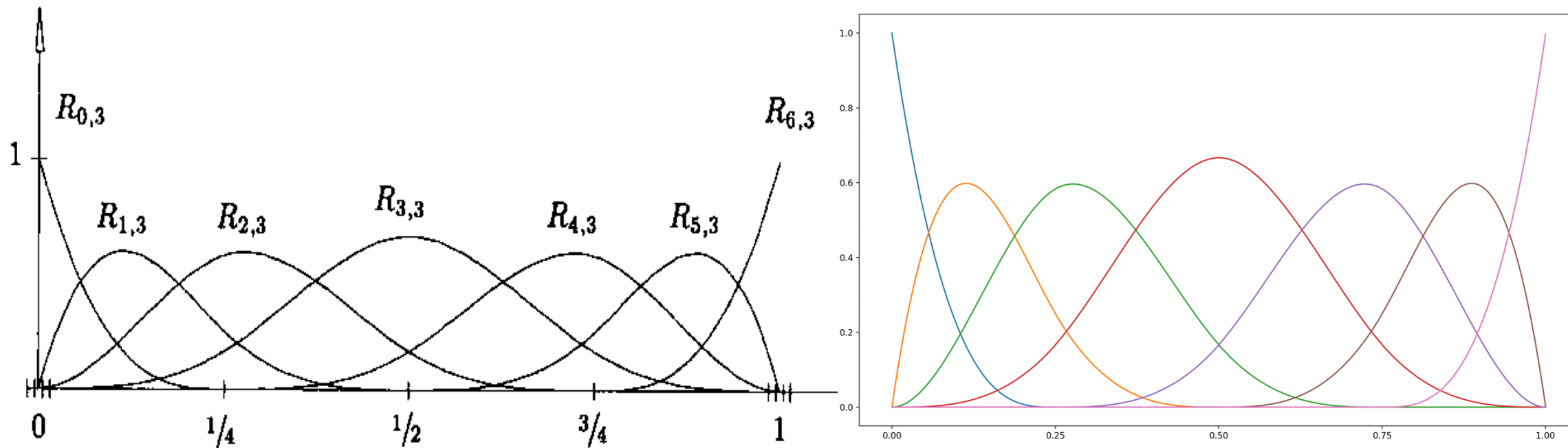


Figure 4.3(a) of *The NURBS Book* by Piegl and Tiller 1997, reproduced in MFEM.

NURBS: NonUniform Rational B-Splines

- Curves:

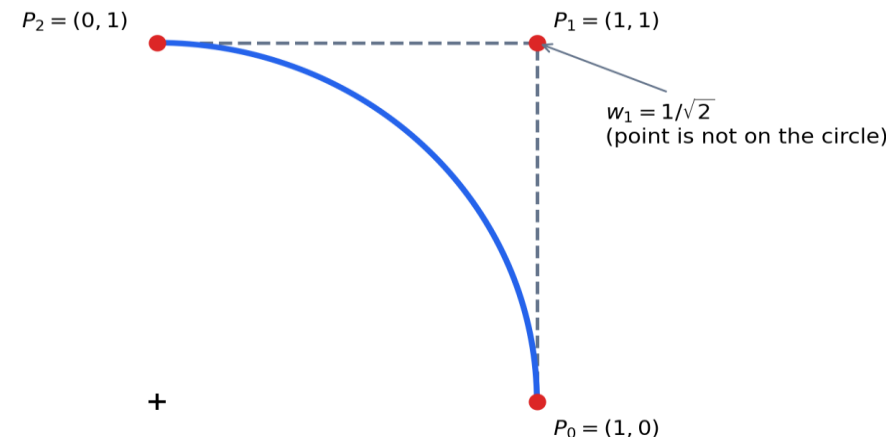
$$\mathbf{C}(u) = \frac{\sum_{i=0}^n N_{i,p}(u) w_i \mathbf{P}_i}{\sum_{i=0}^n N_{i,p}(u) w_i}$$

- Surfaces (tensor-product):

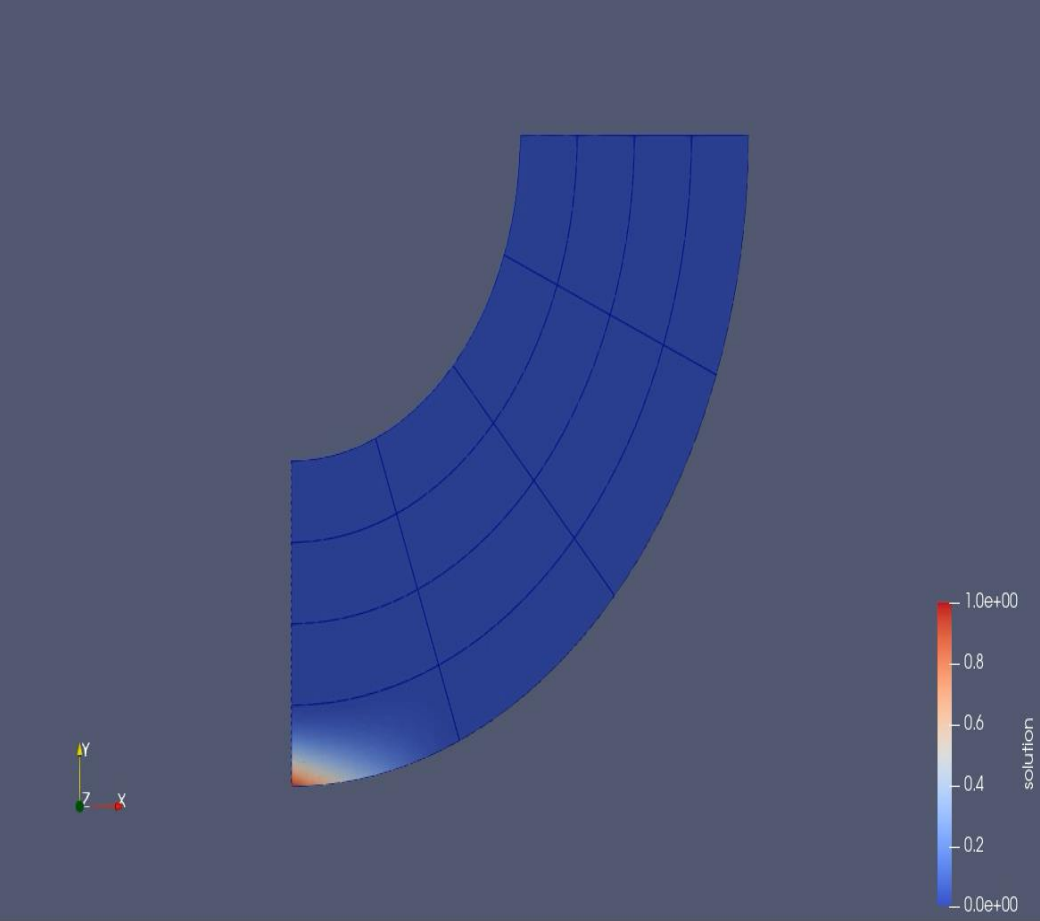
$$\mathbf{S}(u, v) = \frac{\sum_{i=0}^{n_u} \sum_{j=0}^{n_v} N_{i,p}(u) N_{j,q}(v) w_{i,j} \mathbf{P}_{i,j}}{\sum_{i=0}^{n_u} \sum_{j=0}^{n_v} N_{i,p}(u) N_{j,q}(v) w_{i,j}}$$

- B-splines result from unit weights $w = 1$.
- Control points $\mathbf{P}$ guide the curve (not generally interpolated).
- Weights control how strongly each control point attracts the curve.

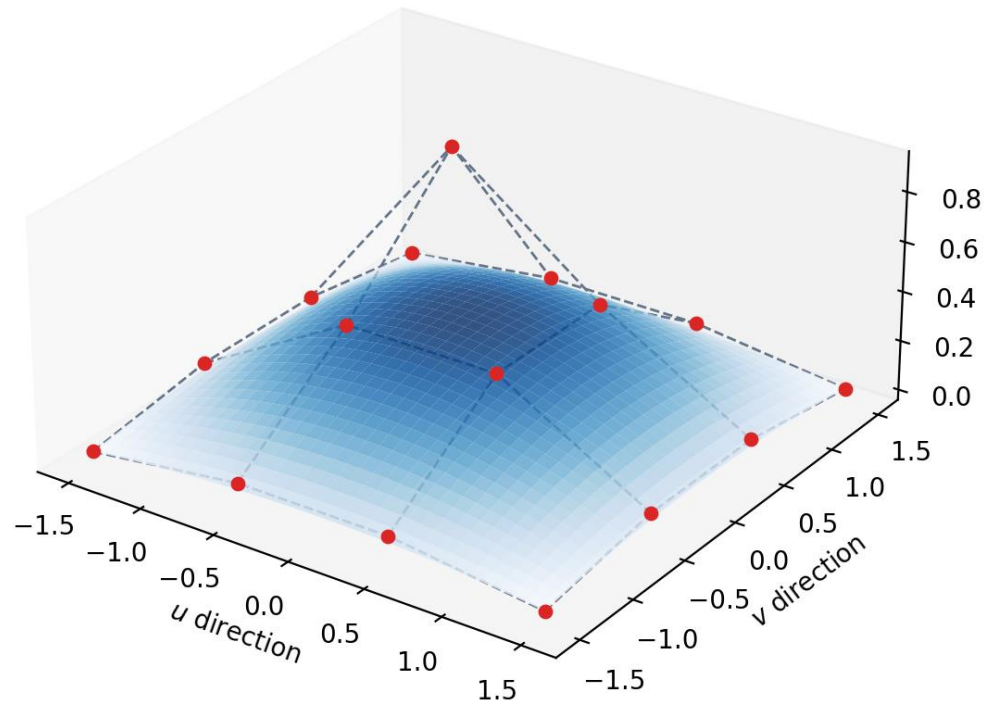
Quadratic NURBS: Exact 90° Circular Arc



NURBS surface patch



Tensor-Product NURBS Patch and Control Net



Examples of basis functions (left) and a surface (right).

NURBS properties

- Basis is a partition of unity.
- Basis functions span multiple elements.
- Elements are knot-spans, and knot insertion gives refinement.
- Continuity is ***at least*** $p - k$ at knots of multiplicity k .
- For $m + 1$ knots and $n + 1$ control points:

$$m = n + p + 1$$

Knots determine basis functions

B-spline basis with 4 elements, degree 3, C^2

- Interior knot multiplicity 1, smoothness $p - 1 = 2$.

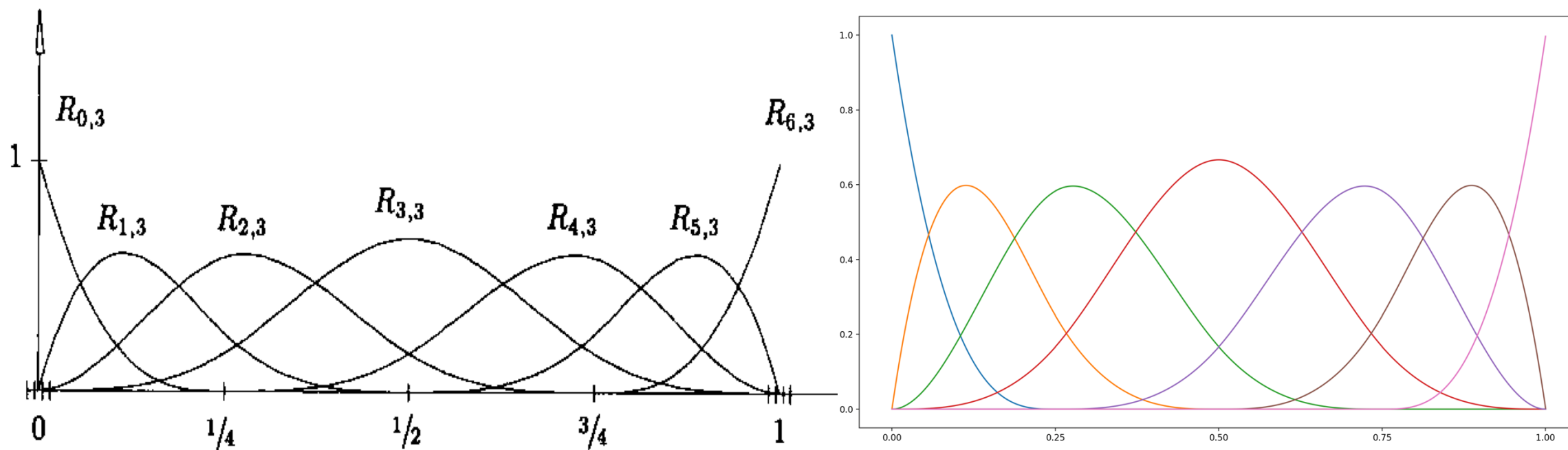
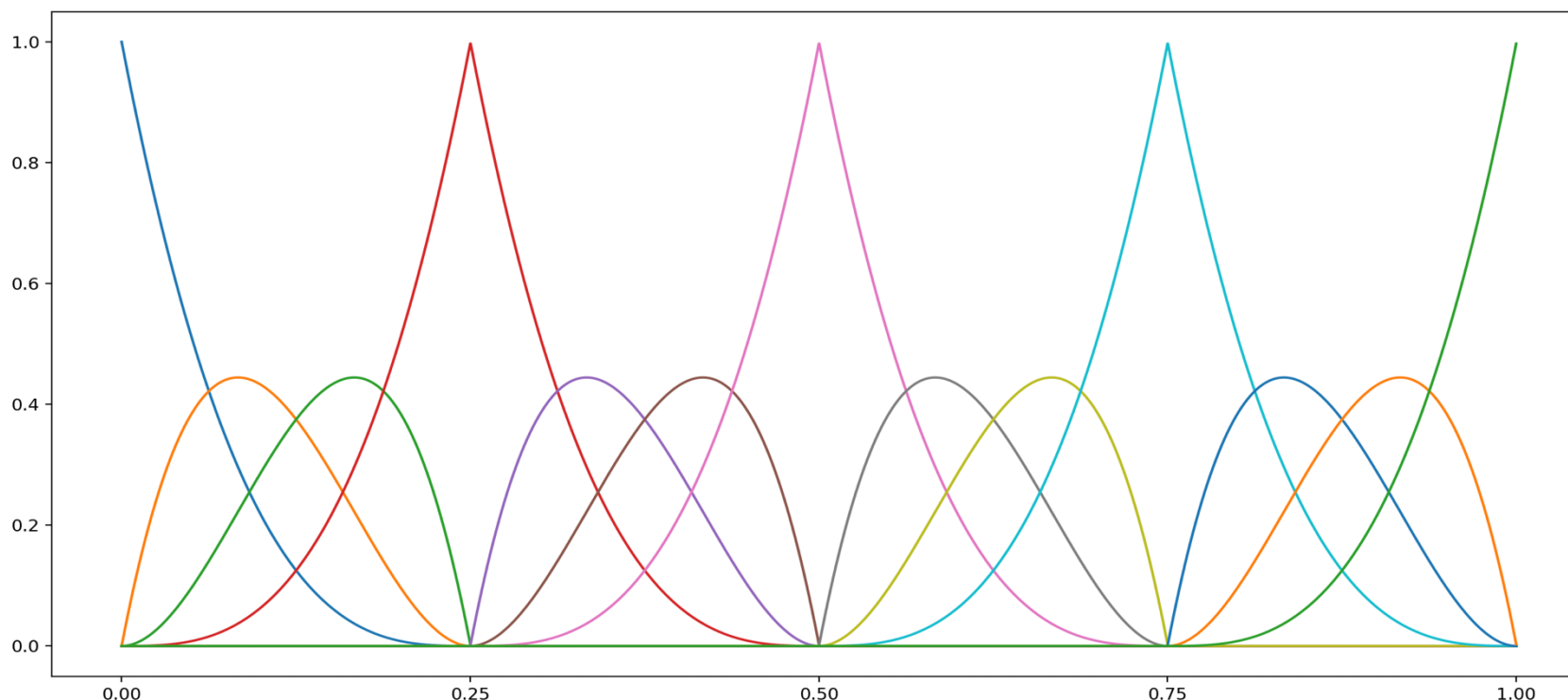


Figure 4.3(a) of The NURBS Book reproduced in MFEM.

B-spline basis with 4 elements, degree 3, C^0

- Interior knot multiplicity 3, smoothness $p - 3 = 0$.



Smoothness decreases with increasing knot multiplicity.

NURBS advantages other than curved geometry

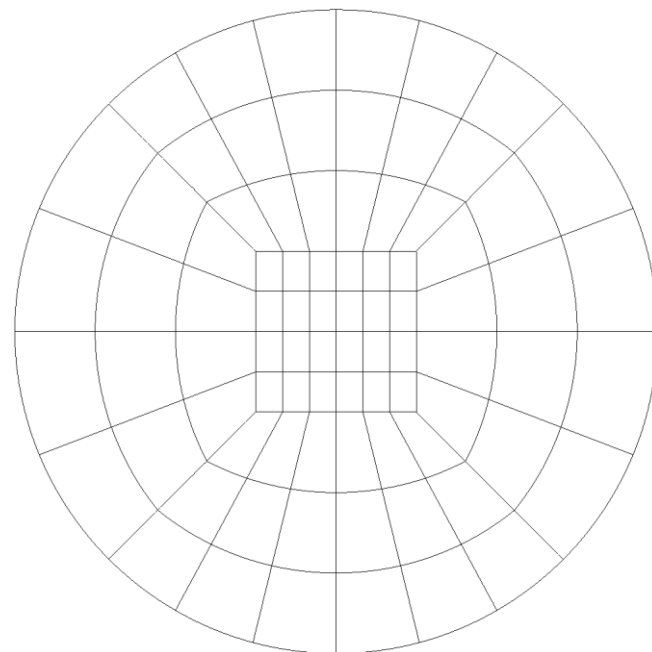
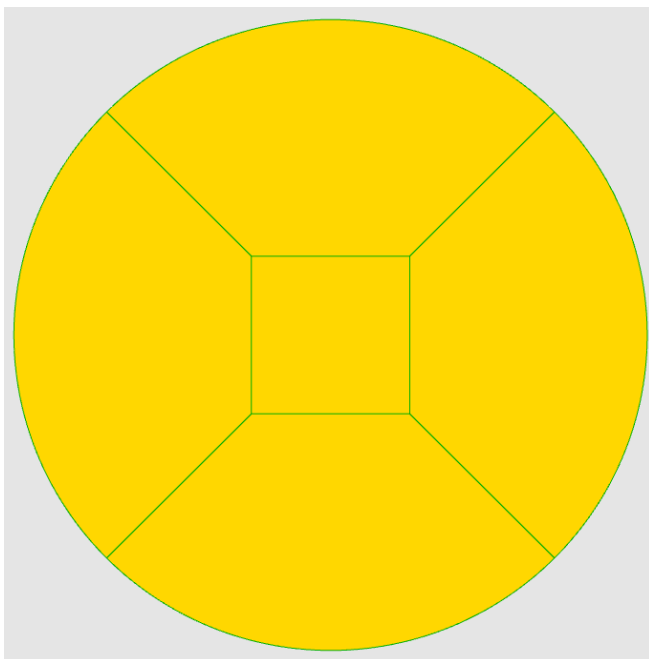
- Block-structured patches
- Element spacing formulas for refinement
- Refinement factors
- Isogeometric analysis
- Minimal coarse mesh



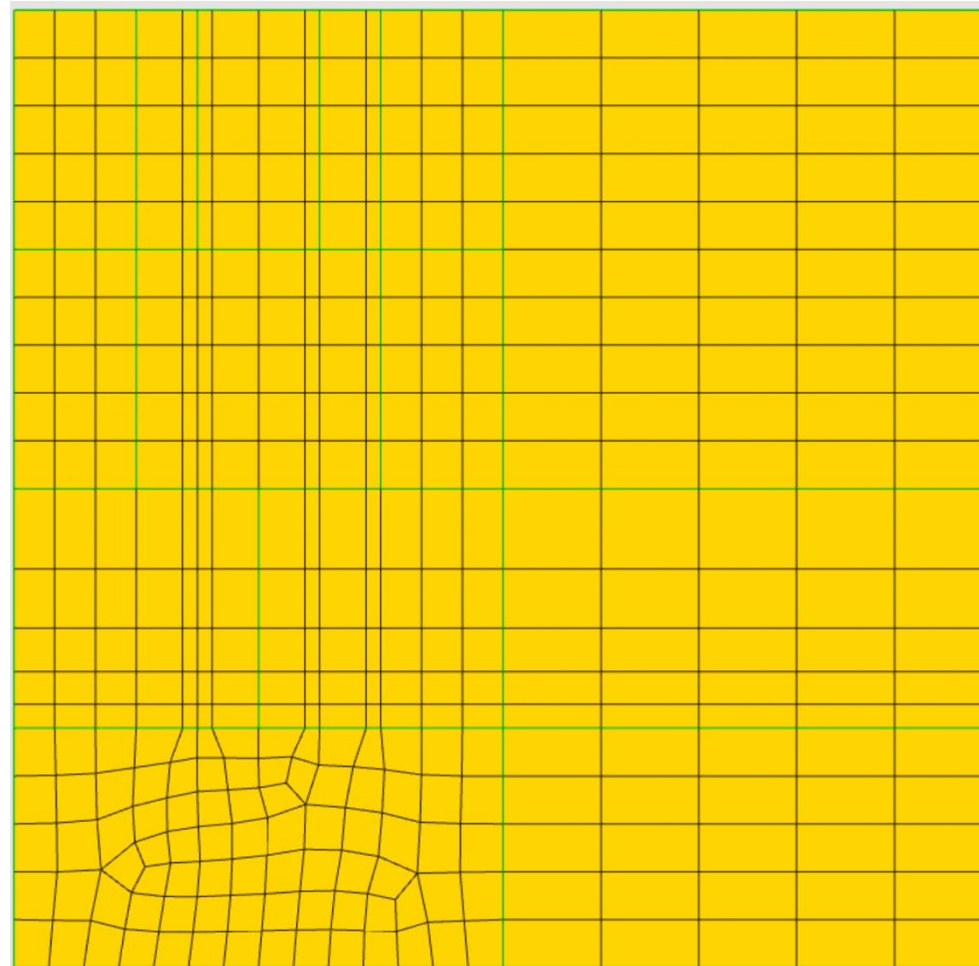
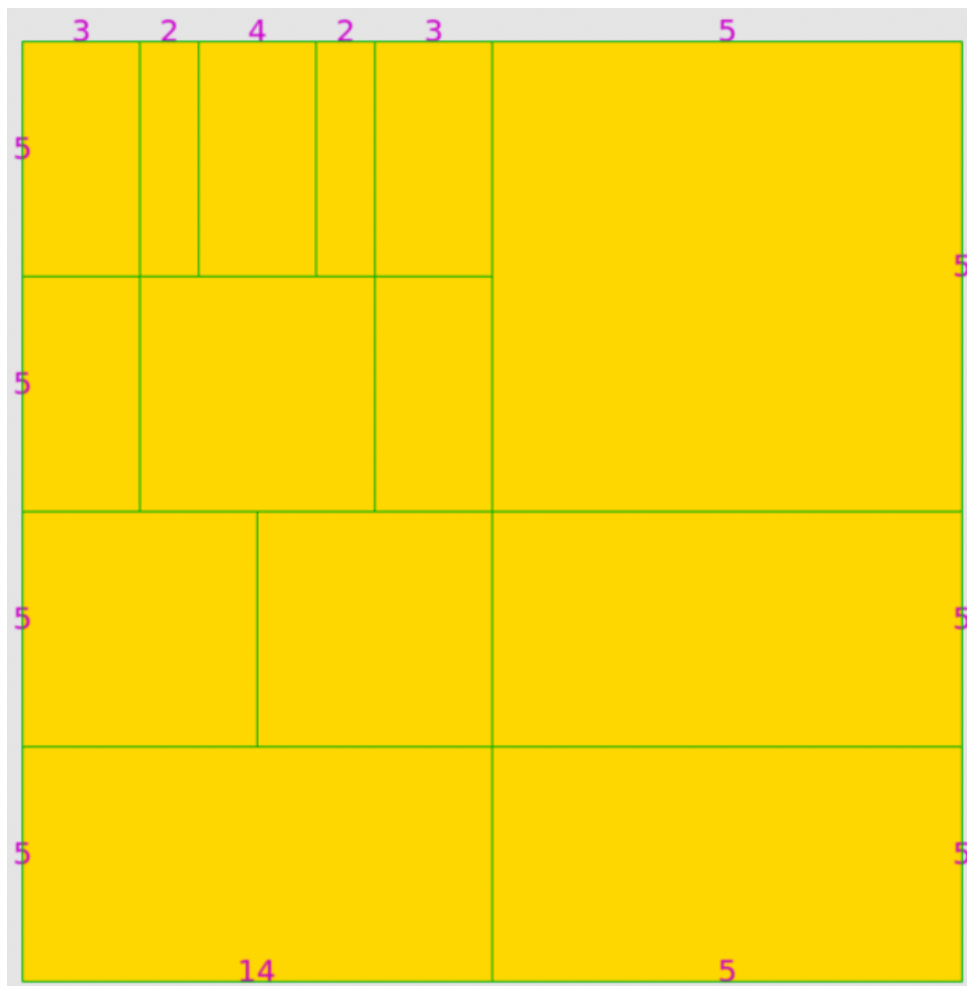
Nonconforming patches

Standard NURBS meshes have conforming patches

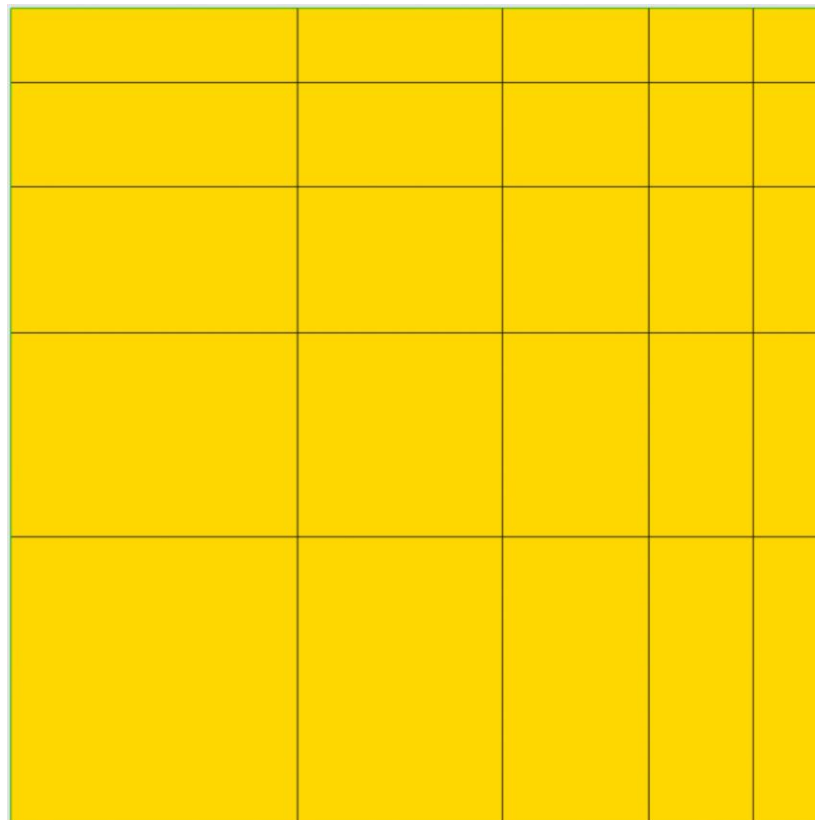
- Continuity on patch interfaces requires matching knots, weights, and control points.
- The patch topology mesh (left) must be conforming.



Challenge: physically-defined patches may not align.



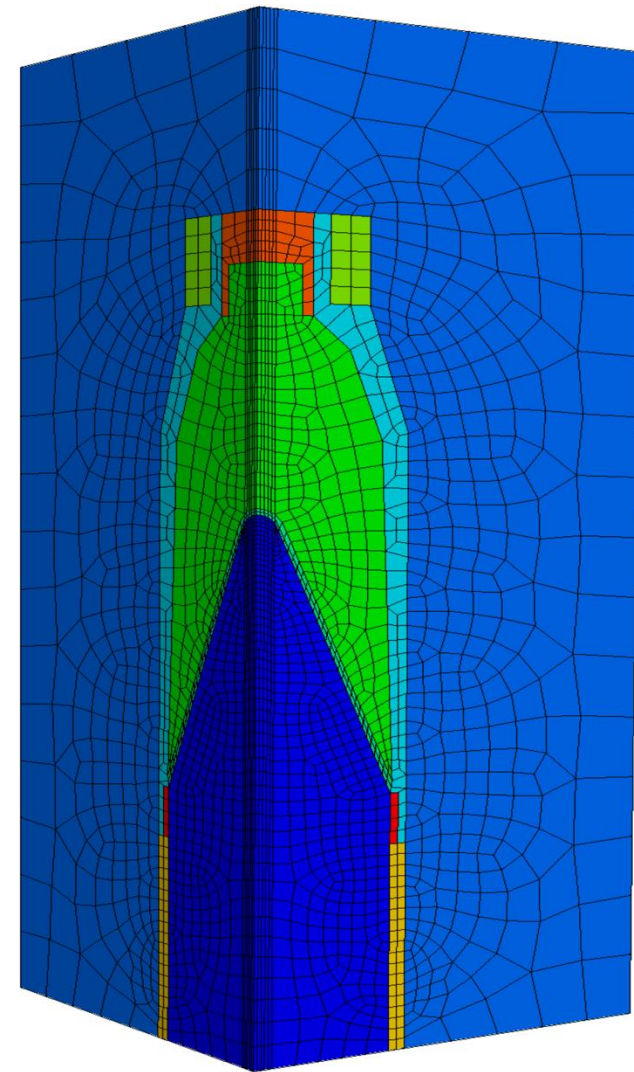
Each patch may have element spacing formulas in each direction



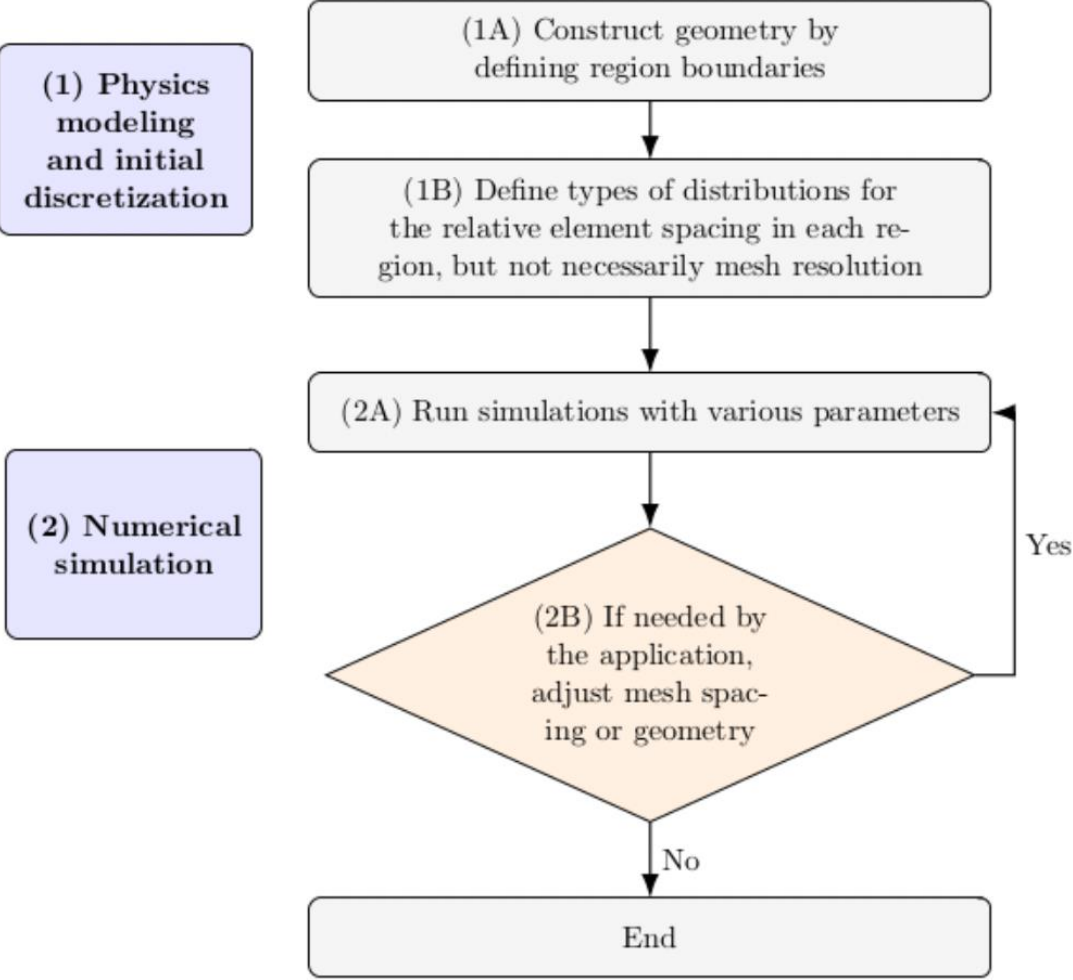
Refinement is done by knot insertion.

Conforming meshes with non-conforming patches

- Conforming meshes are generated, unlike T-splines.
- Patches represent different materials or physics regions.
- Patches are more natural and high-level for users to define than elements.
- The NURBS patch block structure allows for spacing formulas and local refinement factors in specific directions.
- Patches only need alignment of overlapping knot-spans.
- The goal for users is to generate the coarsest mesh possible with the geometry and information to refine as desired, for flexibility and performance.



Coarse mesh in a simulation workflow



Refining from a minimal coarse mesh is efficient and flexible.

NURBS for non-NURBS users

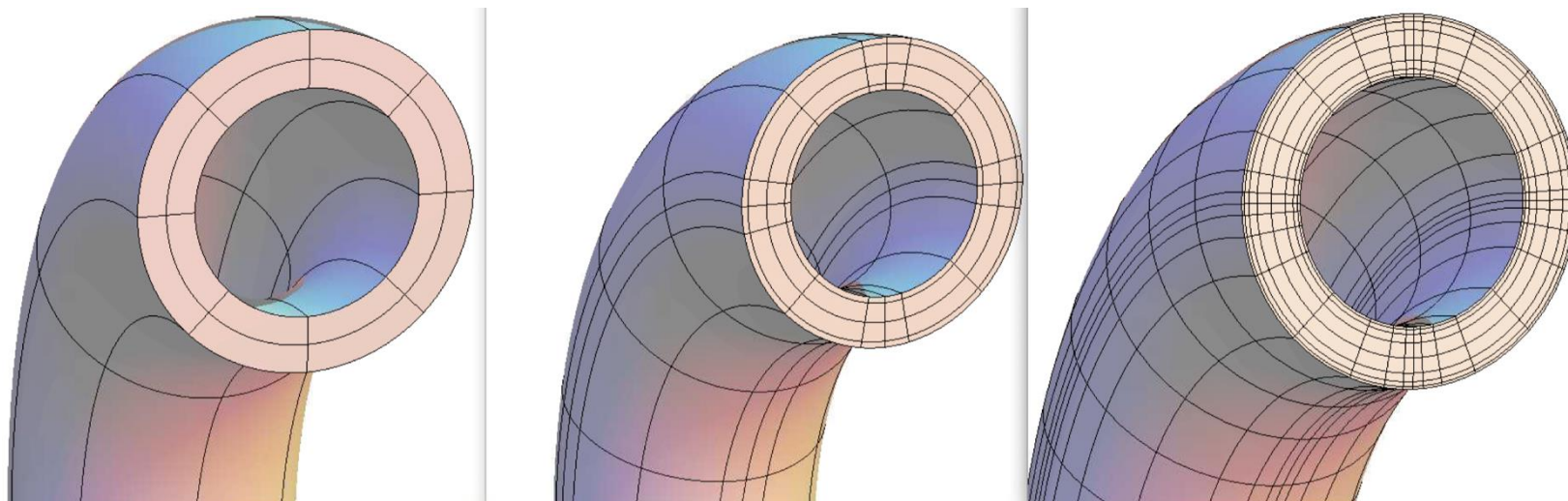
- NURBS meshing can produce standard high-order meshes for non-NURBS applications.
- NURBS patches offer block structure, which does not exist otherwise in MFEM for standard elements in a region.
- Spacing formulas and local refinement factors in specific directions.
- AI agents can work with high-level patch structures more easily than elements.

Everyone can benefit from NURBS meshes!

Spacing formulas

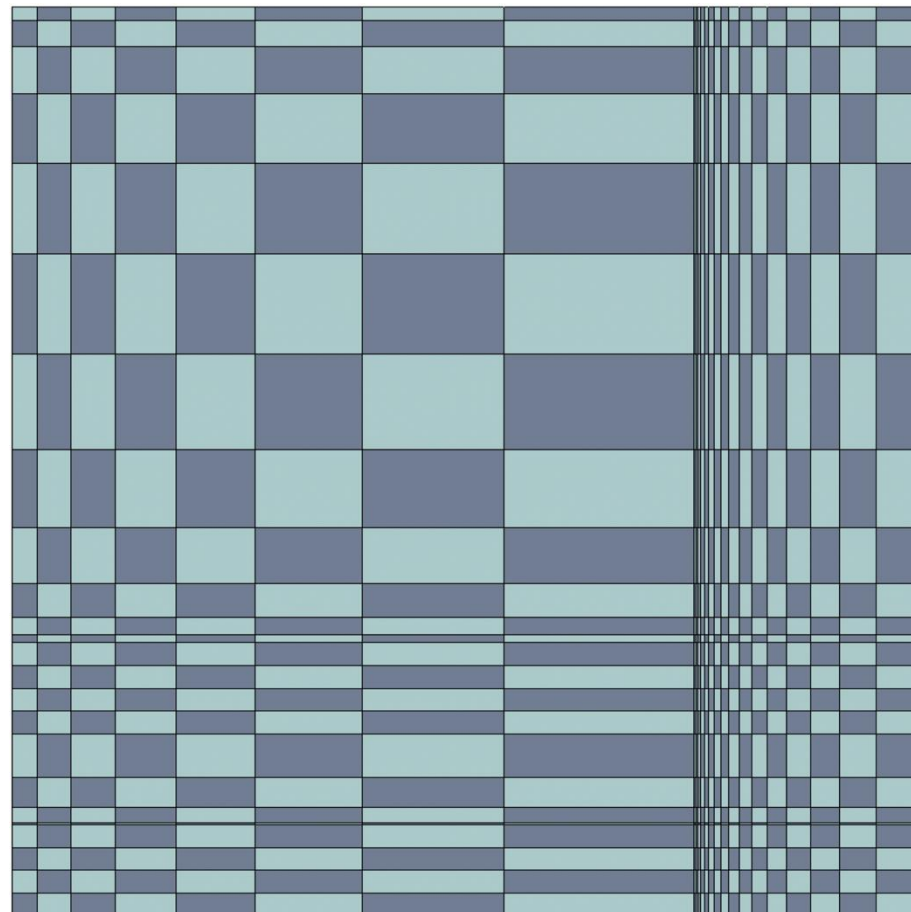
Spacing formulas for refinement

- Each knot vector can have a spacing formula.
- MFEM supports several spacing formula types.
- User-defined formulas can be added: `mesh/spacing.hpp`



Spacing formulas are applied in knot reference space.

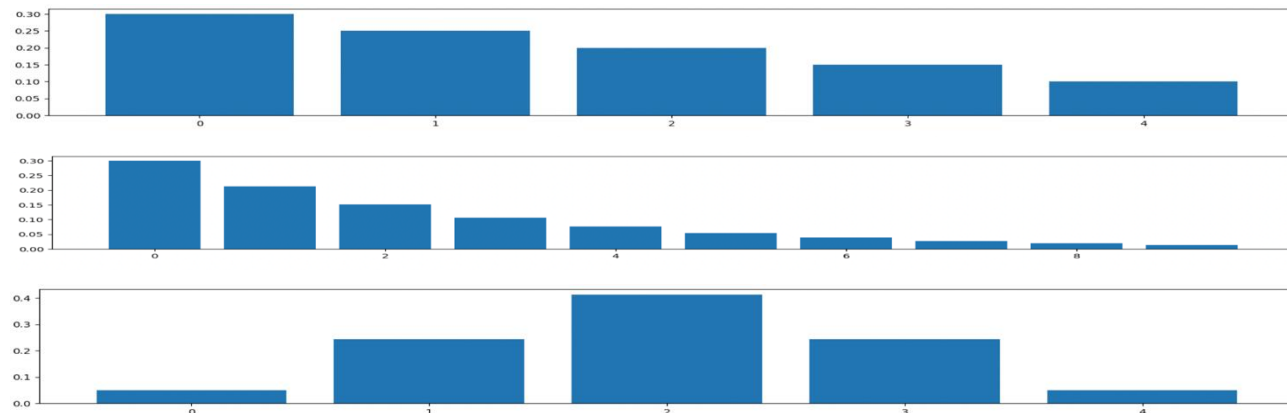
- Single patch: /data/square-nurbs-pw.mesh



Piecewise spacing formulas

Spacing formulas for refinement

- Linear: $s + id$
- Geometric, logarithmic: sr^i
- Bell, Gaussian
- Piecewise: multiple spacing formulas in fixed subintervals
- Partial: a fixed subinterval of a spacing formula
- User-defined: derive from 'class SpacingFunction'.



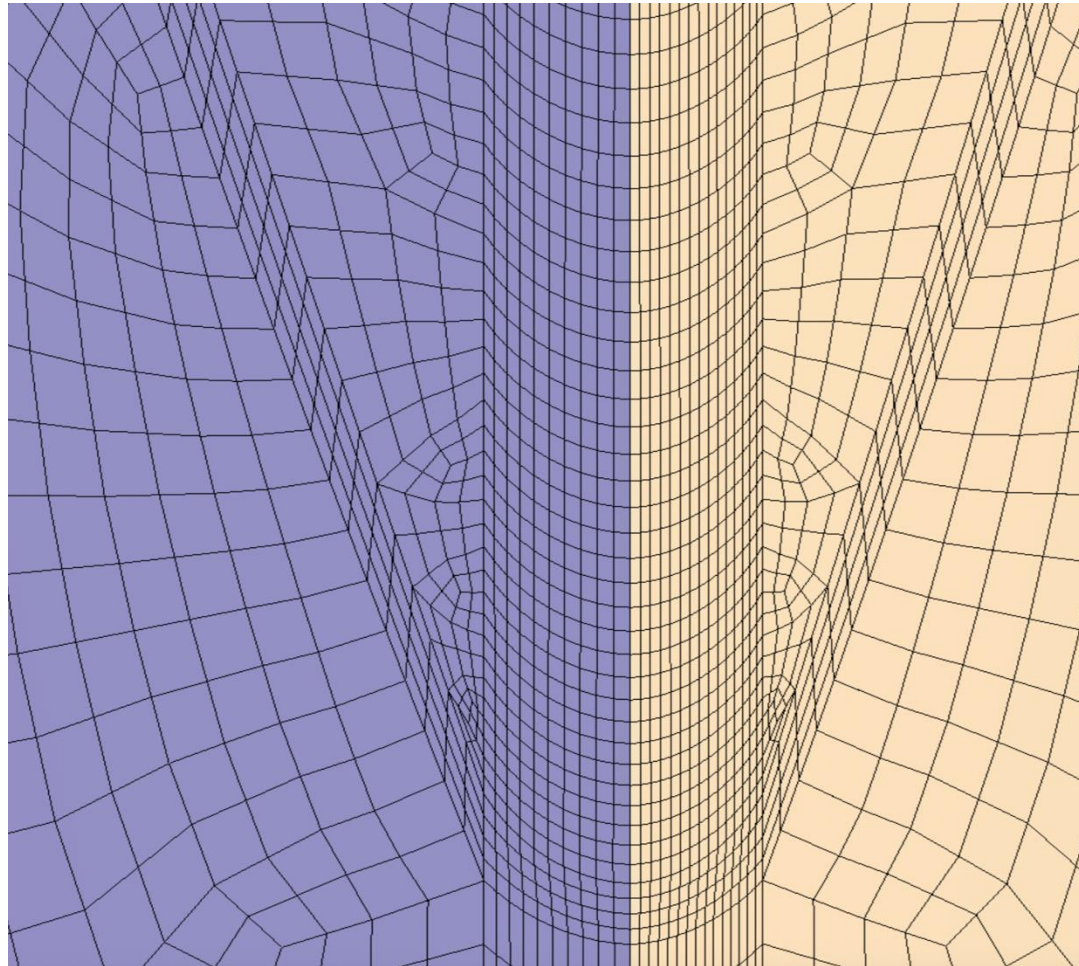
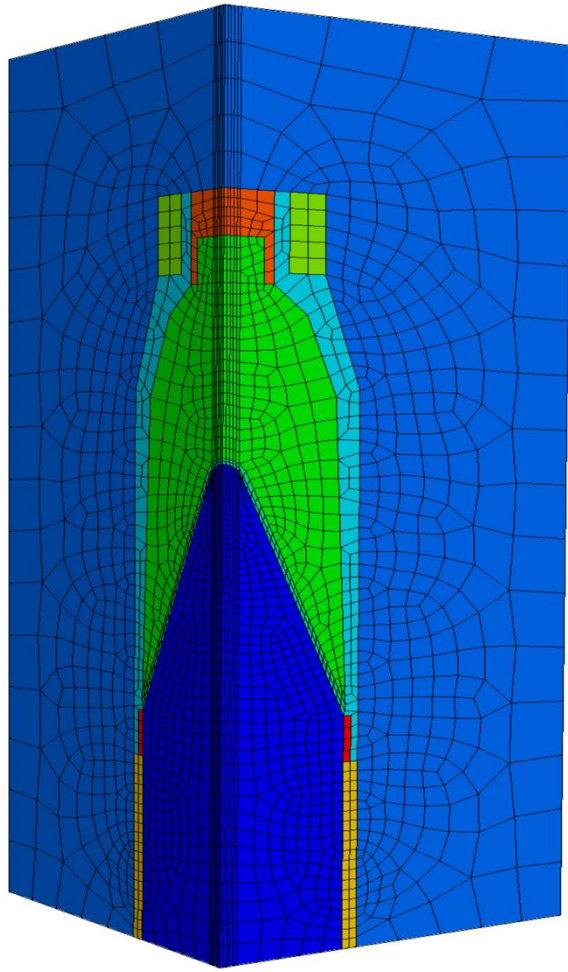
See [mesh/spacing.hpp](#)

Spacing formulas

- Many formulas are non-nested: linear, geometric, Bell, Gaussian.
- Some are nested: uniform, logarithmic.
- Piecewise and partial formulas depend on other formulas.
- Non-nested formulas require full coarsening to a single element.

Refinement by local factors

Local directional refinement factors

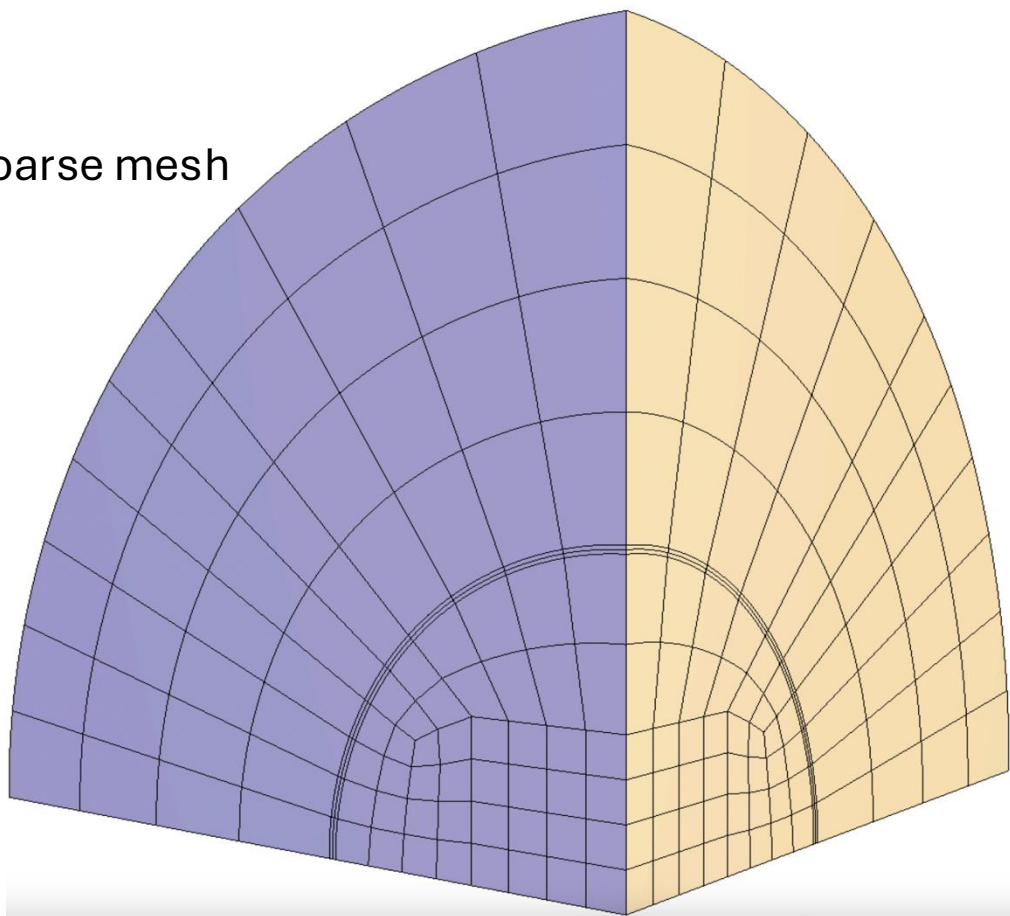


Factors 2 and 3

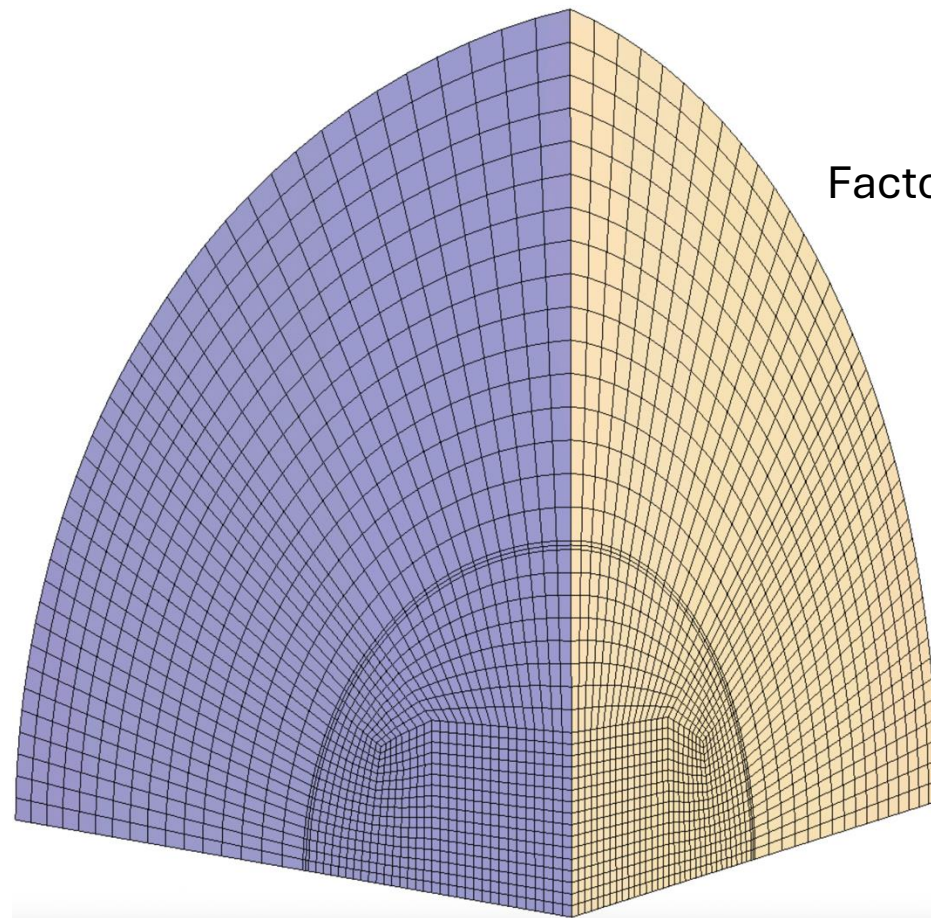
3D shaped charge mesh

Local directional refinement factors

Coarse mesh



Factors 4 and 1



The thin shell has its radial resolution preserved.

Performance benefits

- Meshes with unstructured regions can be expensive to generate at different resolutions.
- Refinement is much faster than mesh generation.
- Mesh quality is preserved through refinement.
- Only 1 coarse mesh needs to be generated.
- Factors can be set for a subset of knot vectors.
- Support is added in the mesh-explorer miniapp.

Refinement option 'p' in mesh-explorer miniapp

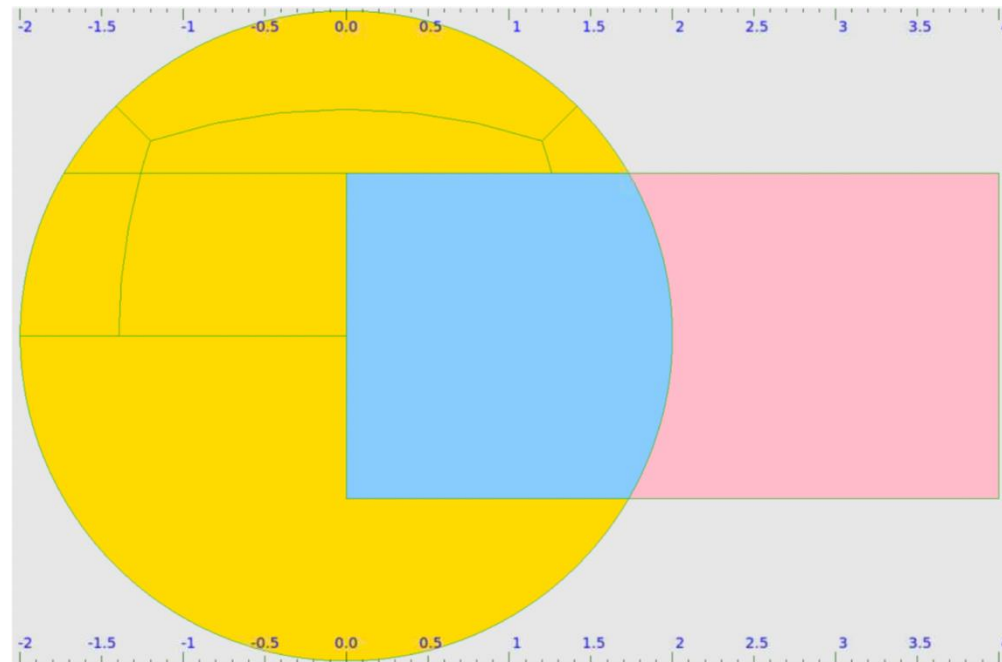
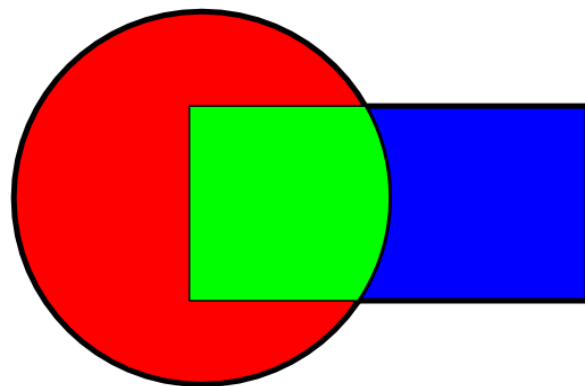
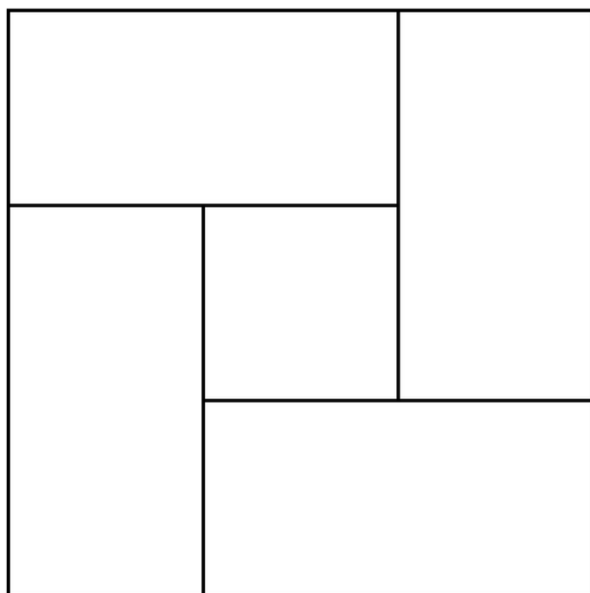
- Refinement factors can be set for a subset of knot vector indices (and a default factor for all other knot vectors).
- Knot vector factor file format:
number of lines to follow
kv_index num_factors=1 factor
- Only 1 factor per knot vector is supported so far.
- Example:
1
5 1 2



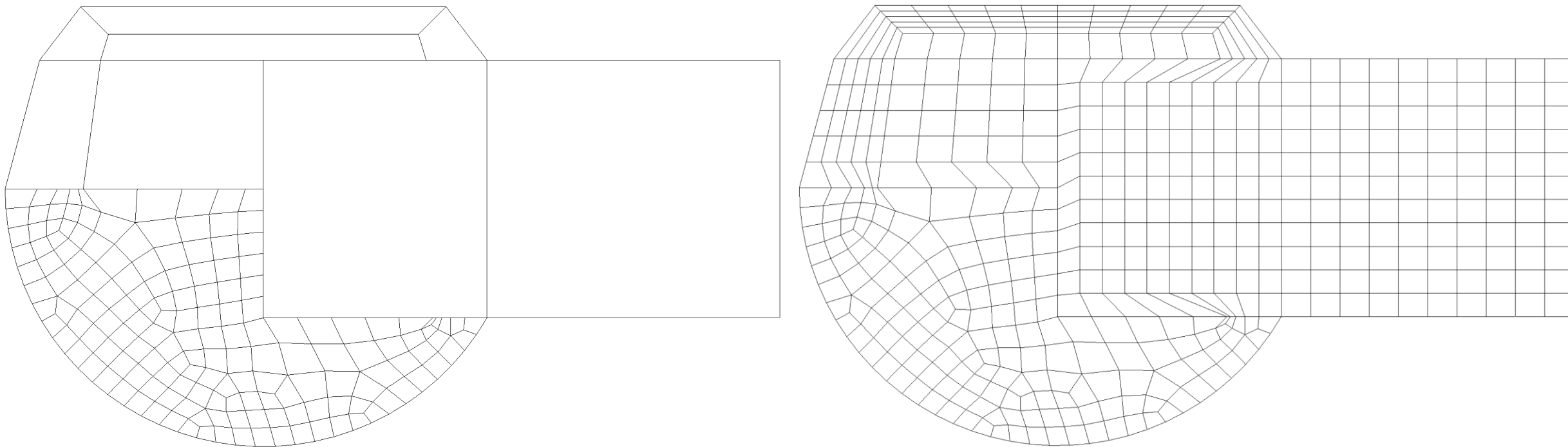
NC-patch mesh generation

Patch topology NCMesh is very flexible

- A standard element NCMesh for H^1 requires a hanging vertex or edge to interpolate its master edge or face.
- For the patch topology mesh, no such restrictions are needed.
- *Auxiliary* edges and faces are possible.

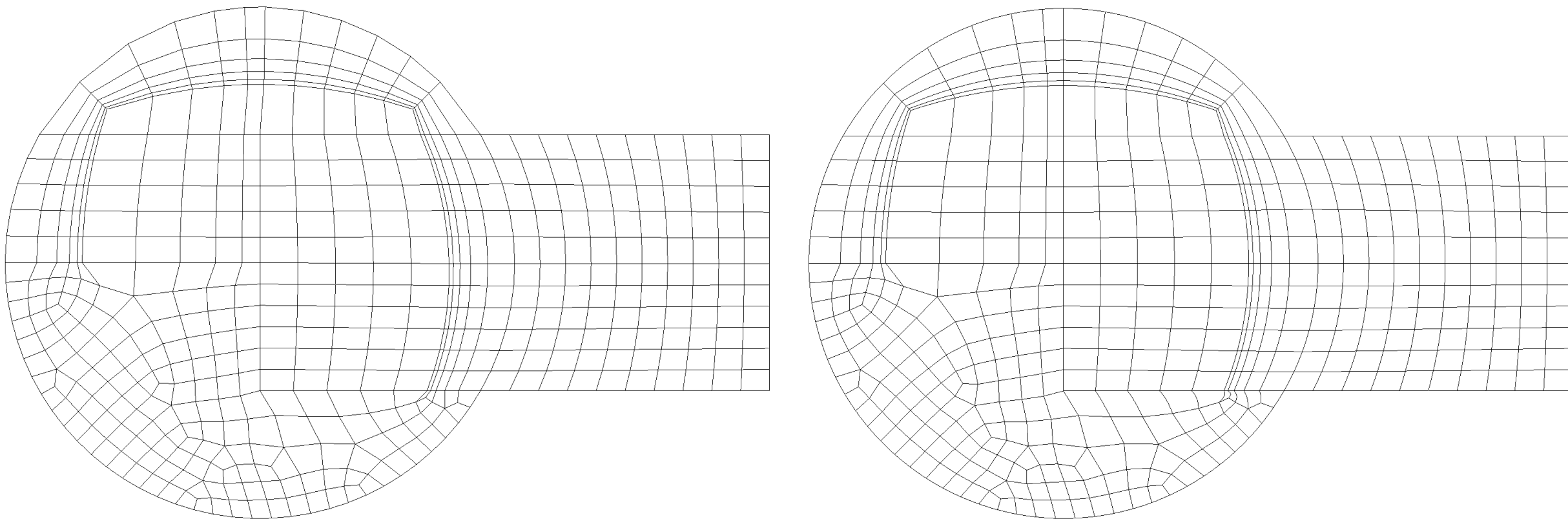


Construction of NC-patch meshes



Patch topology (left) and initial linear mesh (right).

Construction of NC-patch meshes



Linear mesh (left) and high-order NURBS mesh (right).

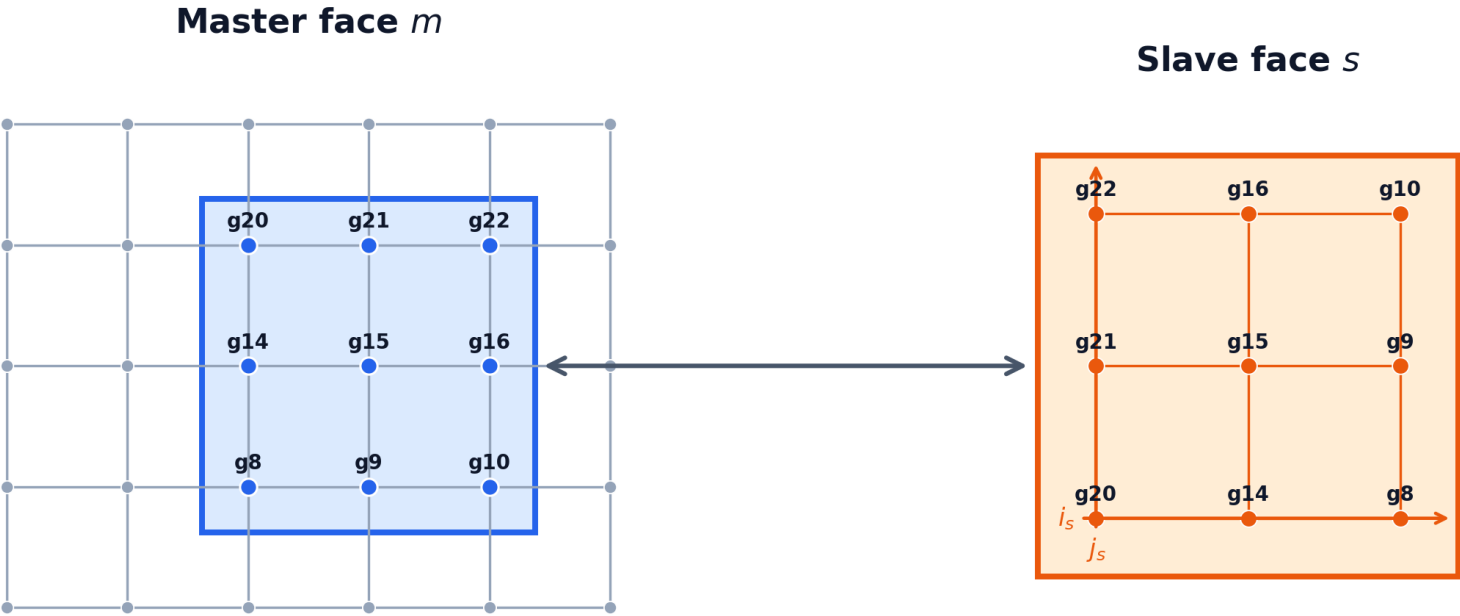
Construction of NC-patch meshes

1. Construct the linear patch topology mesh.
2. Determine consistent numbers of elements in each direction on each patch, using equivalence classes of edges.
3. Determine valid orientation of patches and reorder or reorient them. Also map the patch DOF indices.
4. Construct a linear NC-patch NURBS mesh with uniform spacing and the correct number of elements in each patch.
5. Modify the existing linear NC-patch NURBS mesh to have correct vertex positions and spacing formulas.
6. Elevate the linear NC-patch NURBS mesh to the desired degree.
7. Set control points for the prescribed degree using a DOF map on each patch.

MFEM NURBS NC-patch mesh v1.0

Continuity of NC-patch meshes

- Continuity is enforced by identifying control points shared by nonconforming edges and faces.

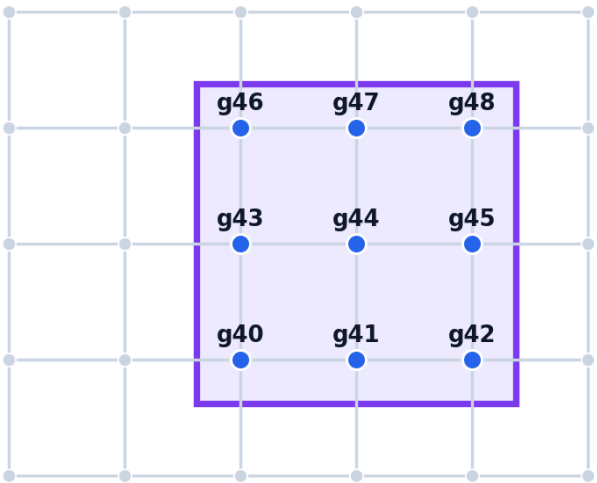


Shared face DOFs are identified by global indices.

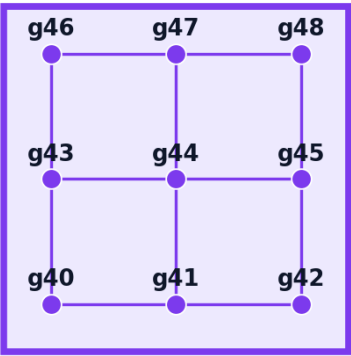
Continuity of NC-patch meshes

- Continuity is enforced by identifying control points shared by nonconforming edges and faces.

Master face m_0 (canonical parent)

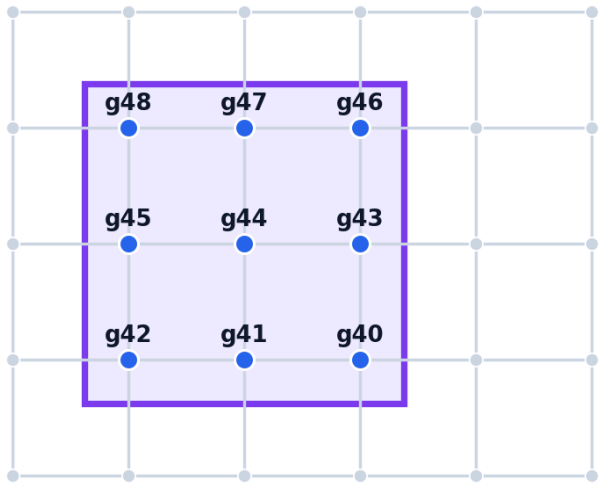


Auxiliary face a



one ordering retained from the canonical parent

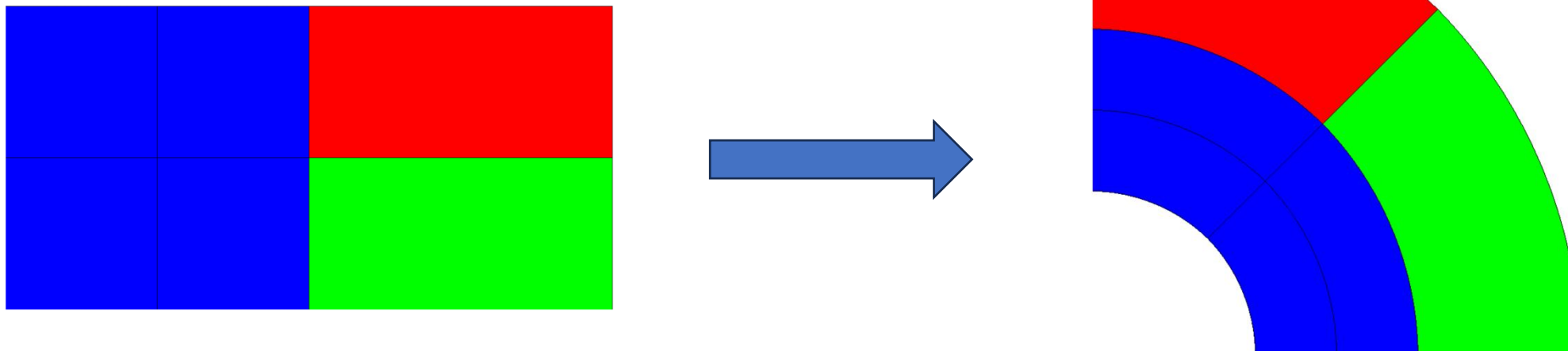
Overlapping master face m_1



Auxiliary faces may have multiple master faces to match.

Continuity of NC-patch meshes

- Knot multiplicities are increased on master entities to match the number of control points at hanging vertices with C^0 continuity.
- Geometric accuracy does not degrade.
- Circle radius error is machine precision.



- Hanging vertices have C^0 continuity

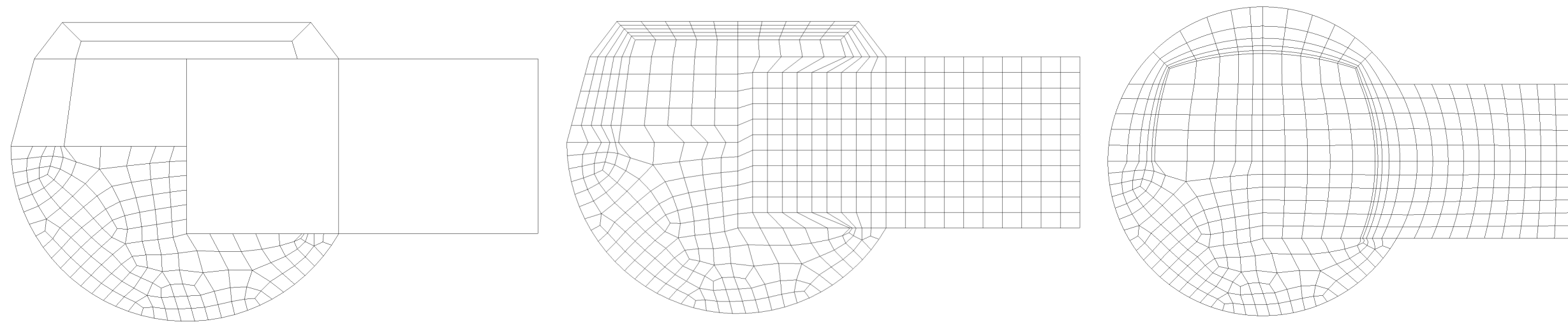
NURBS mesh generation tools

- PMESH (not open-source) was used to generate examples.
- splinepy is open-source: <https://github.com/isosuite/splinepy>
- Generating individual patches is the first step.
- Assembling patches consistently for a global mesh is the main challenge.
- AI for mesh construction or to generate a generation miniapp?

Opportunity: create a new open-source MFEM mesh generator!

Algorithm -> AI skills file

- AI can be prompted to follow the steps of the algorithm.
- Each step can be an iterative human + AI process.
- NC-patch NURBS meshes are complicated only for humans.

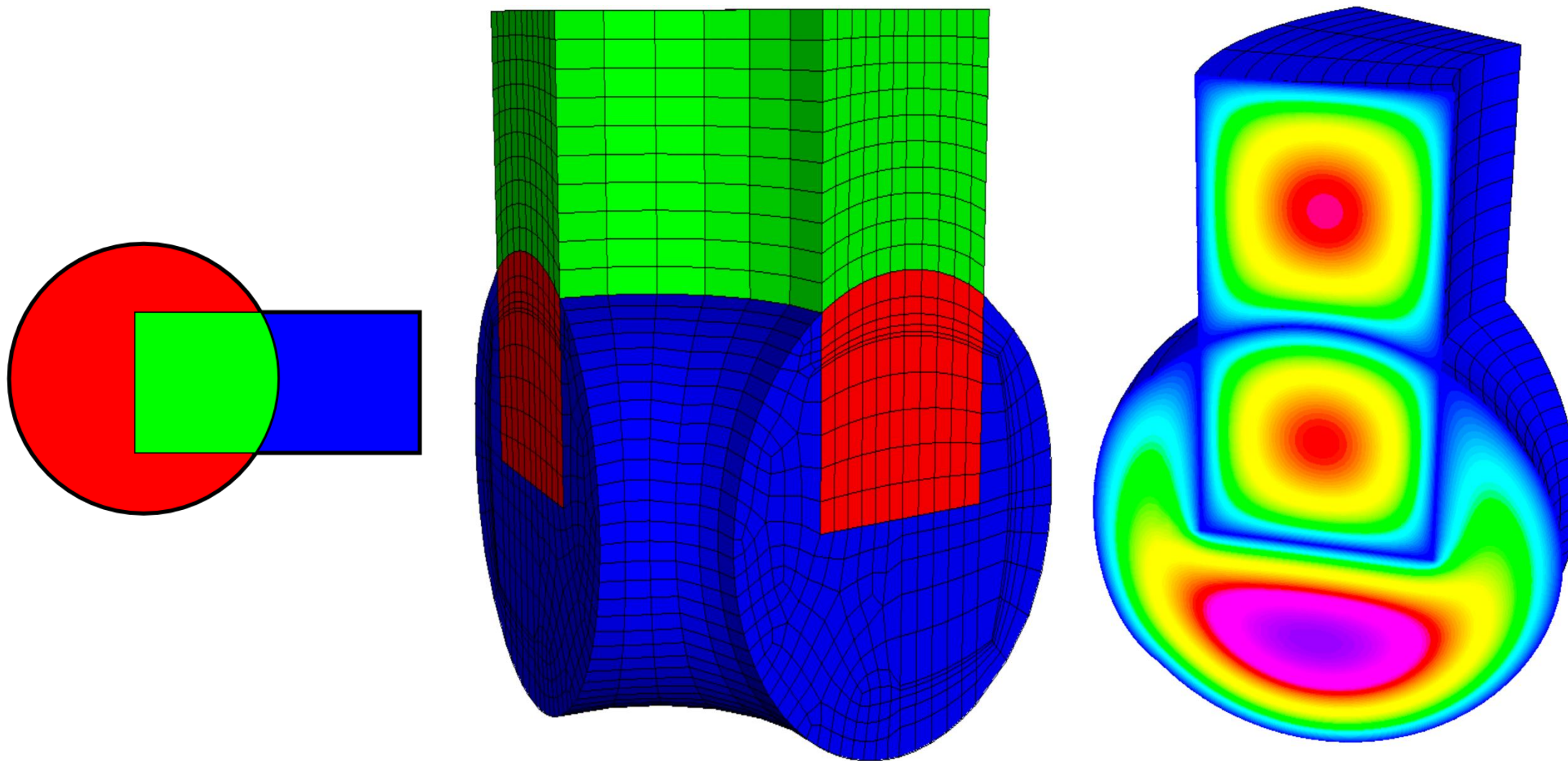


Meshes can be generated without a generator!

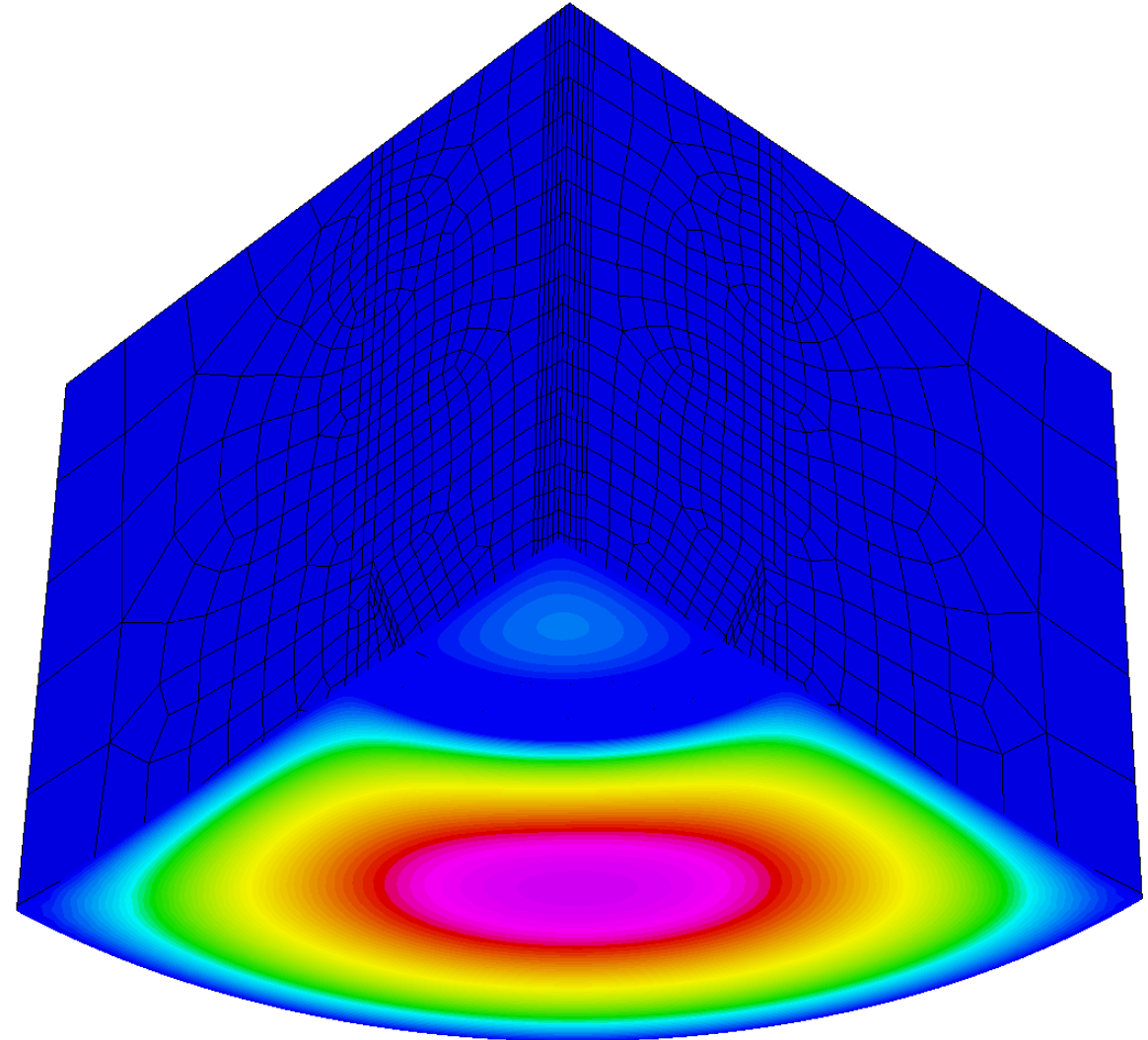
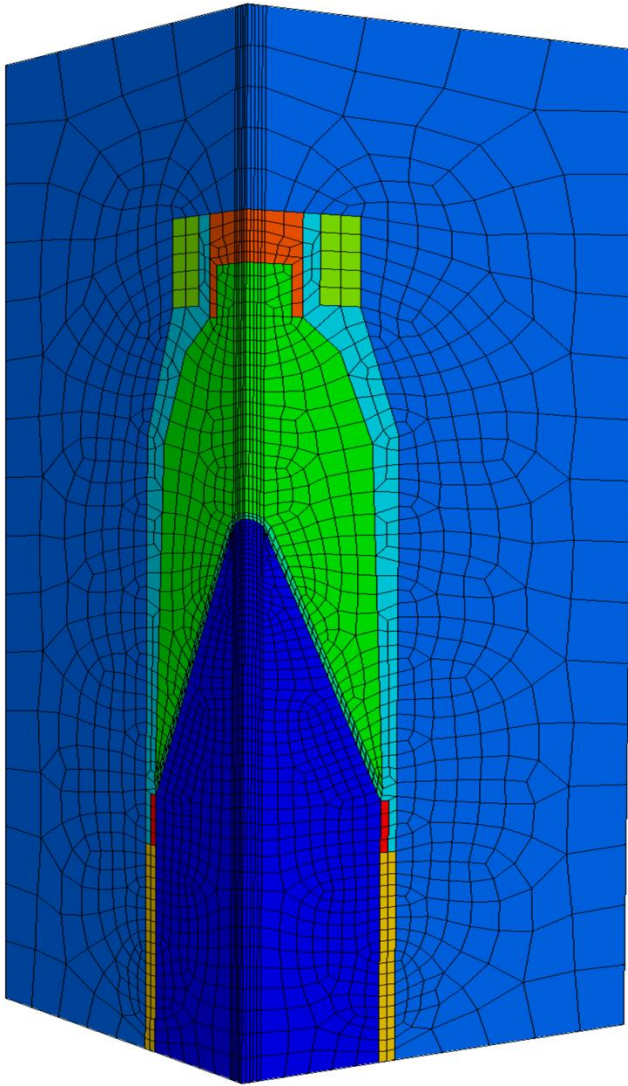


Examples

IGA diffusion solver on revolved Schwarz domain



IGA diffusion solver on shaped-charge domain



Conclusions

- A more general NURBS format is enabled, with NC patches.
- Capabilities:
 - local refinement factors
 - spacing formulas
 - more freedom to define physical regions
 - high-level control (for users and AI agents).
- Performance benefits:
 - fast, robust refinement
 - only one coarse mesh generation.

Thank you for your attention!
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