Numerical Solvers for Viscous Contact Problems in Glaciology

Gonzalo Gonzalez de Diego

Photograph: Kverkfjöll Glacier Cave, Iceland (from worldlandforms.com)

NYU

About me

PhD – Ice sheets

Photo: NASA Operation IceBridge

I. Hewitt

P. Farrell

Postdoc – Sea ice

Photo: Stephen Alvarez/Getty

G. Stadler

Mathematical modelling of ice sheets

Ice sheets are large masses of ice that cover most of Antarctica and Greenland. Together, these ice sheets hold a volume of ice that would raise the sea level by 65m if they were to melt. (Farinotti et al. 2019)

Understanding ice sheet dynamics requires a combined knowledge of many processes:

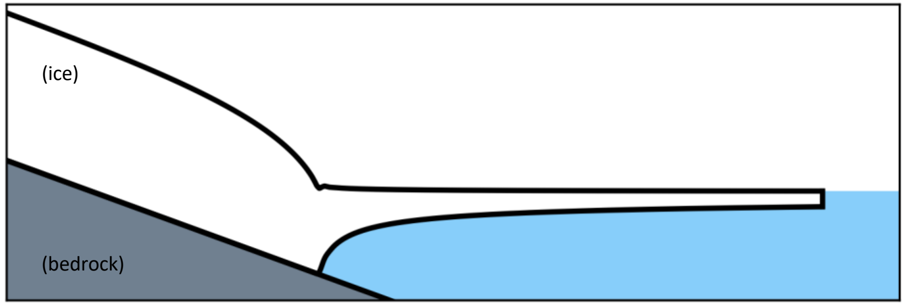
- how it **melts**,
- how ice **flows**,
- how it **slides** over its bedrock,
- how **meltwater** affects the dynamics of the ice sheets,
- …

Elephant Foot Glacier, NE Greenland
(Copernicus Sentinel-2A data, August 2016)

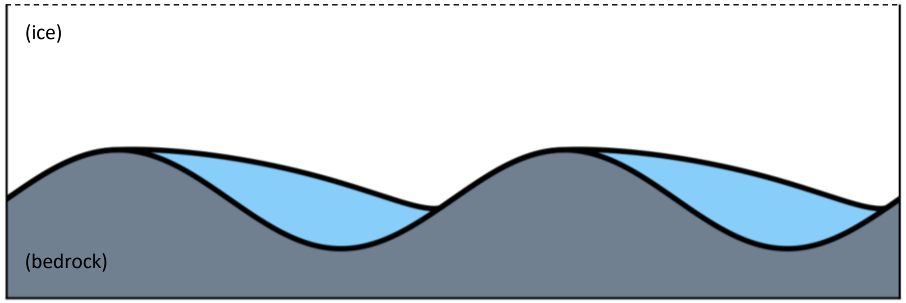
Viscous contact problems in glaciology

In glaciology one encounters **viscous contact problems**. These are time-dependent viscous flow problems where the fluid is in contact with a solid surface from which it can detach and reattach.

Two examples in glaciology:



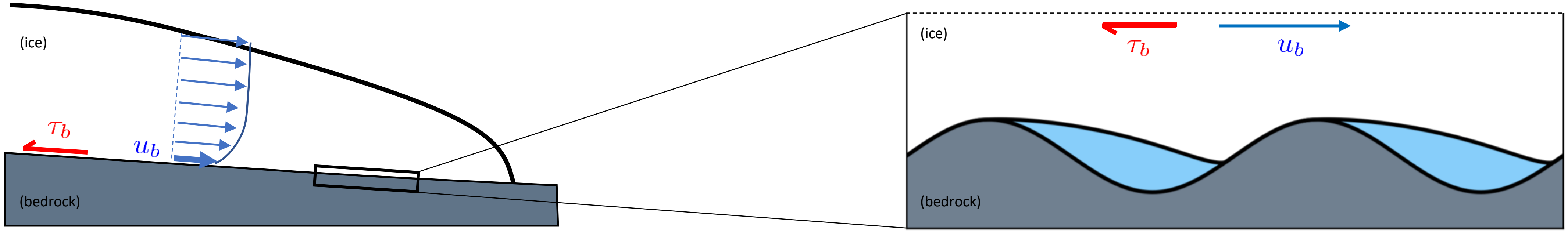
Marine ice sheets



Subglacial cavitation

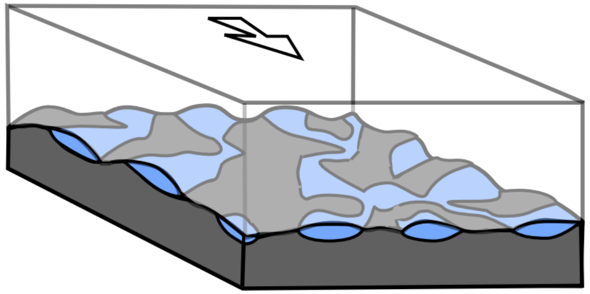
Subglacial cavitation

At the ice-bedrock interface, cavities may form along obstacles due to **high water pressures** and **fast sliding velocities**.



Why is subglacial cavitation important?

- Fundamental mechanism in **glacier sliding** → $\tau_b = f(u_b)$,
- Linked cavities establish a **subglacial hydrology** system.

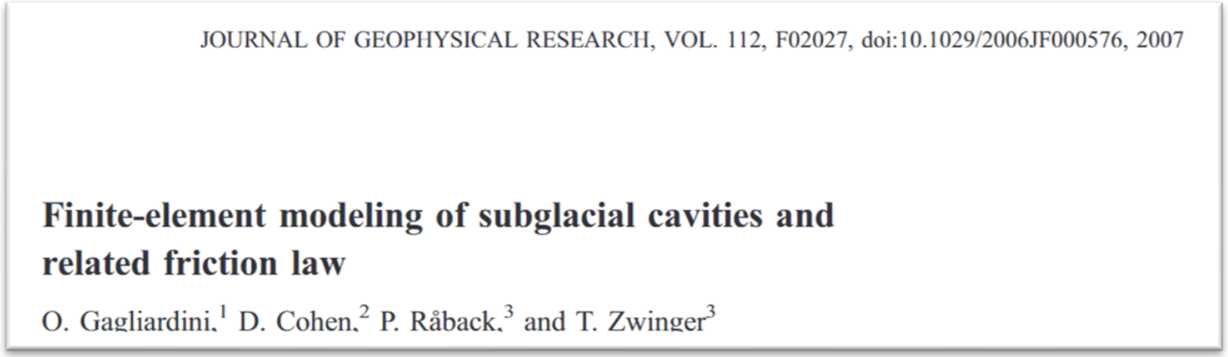


(Figure from I. Hewitt)

Subglacial cavitation

Goal: Develop a robust and accurate numerical scheme for viscous contact problems.

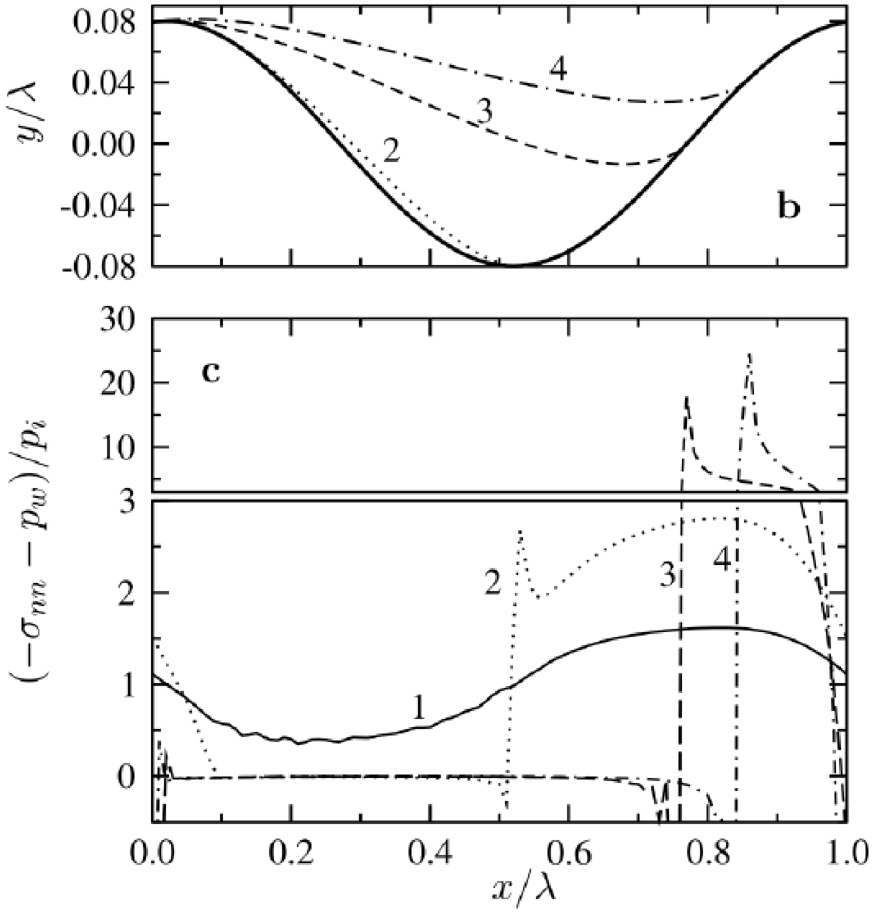
Motivation: Only other computational results with **Elmer/Ice**:


JOURNAL OF GEOPHYSICAL RESEARCH, VOL. 112, F02027, doi:10.1029/2006JF000576, 2007

Finite-element modeling of subglacial cavities and related friction law

O. Gagliardini,[1] D. Cohen,[2] P. Råback,[3] and T. Zwinger[3]

- No recognition of **variational inequality** structure.
- Numerical computation of stresses appear inaccurate and unstable.



(Gagliardini et al. 2007)

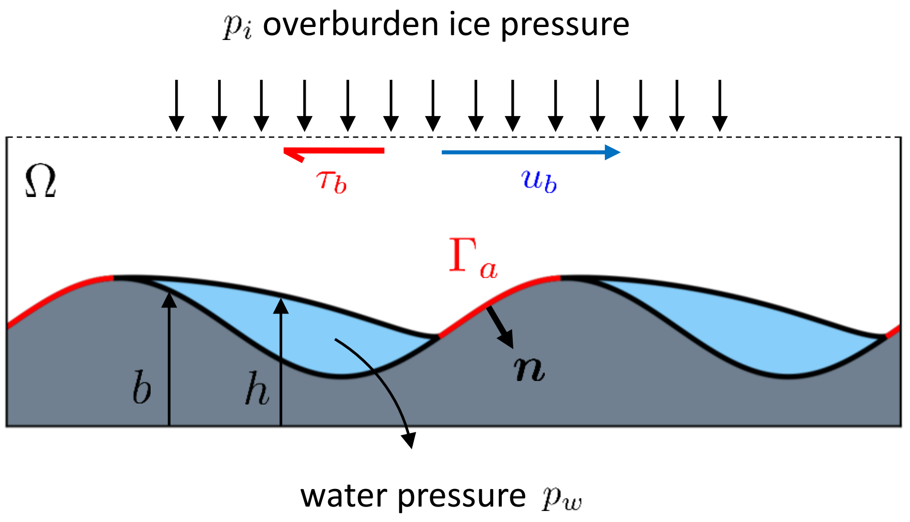
Mathematical modelling of subglacial cavitation

We model the formation of subglacial cavities by solving the **subglacial boundary layer problem** in two dimensions:

Stokes equations with **contact boundary conditions**:

$$\left.\begin{aligned} -\nabla\cdot\left(\alpha|\mathbf{D}\boldsymbol{u}|^{r-2}\mathbf{D}\boldsymbol{u}\right)+\nabla p &= \boldsymbol{f} \\ \nabla\cdot\boldsymbol{u} &= 0 \end{aligned}\right. \quad \text{in } \Omega(t)\,,$$

$$\left.\begin{aligned} \boldsymbol{u}\cdot\boldsymbol{n} &\leq 0 \\ \sigma_{nn} &\leq -p_w \\ (\boldsymbol{u}\cdot\boldsymbol{n})\,(\sigma_{nn}+p_w) &= 0 \end{aligned}\right. \quad \text{on } \Gamma_a(t).$$



Free boundary equation:

$$\begin{aligned} \partial_t h &= -\left[1+(\partial_x h)^2\right]^{-1/2}\boldsymbol{u}\cdot\boldsymbol{n}, \\ h &\geq b. \end{aligned}$$

Numerical considerations

Sketch of our algorithm

Set initial geometry Ω^0

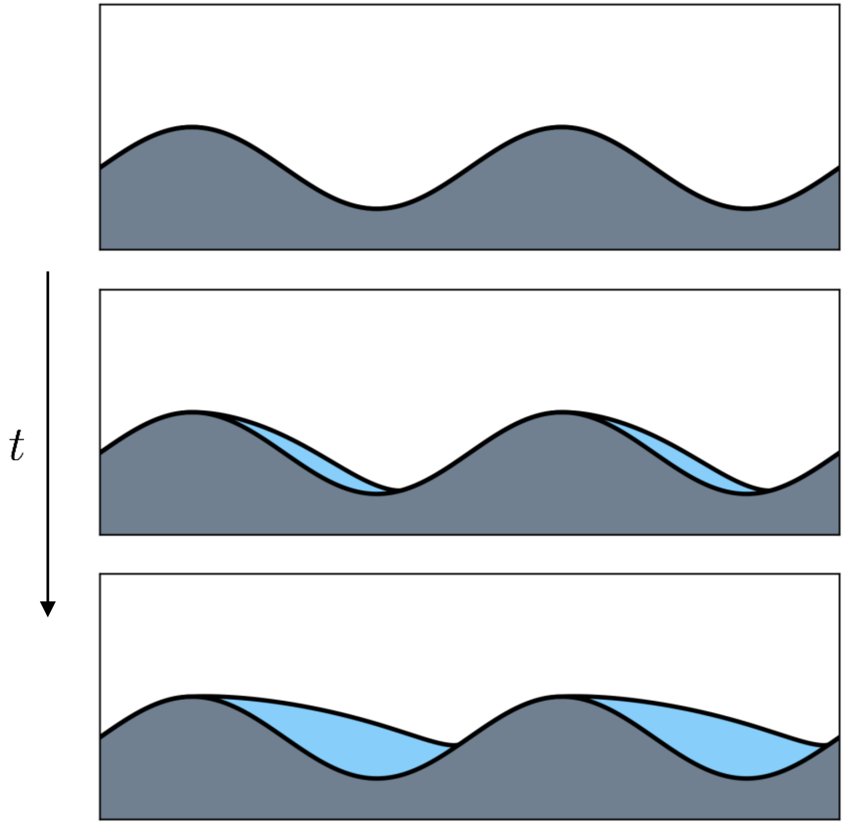
for k = 0,1,2,… do

Solve Stokes equations on $\Omega^k \to \boldsymbol{u}^k$.

Solve free boundary equation with $\boldsymbol{u}^k$.

Deform domain $\to \Omega^{k+1}$.

end



Next slides:

- Solving the Stokes equations with contact boundary conditions.
- Coupling the Stokes equations to the free boundary equation.

Numerical considerations

Weak formulation – The PDE can be rewritten as a **variational inequality.**

Find $(\boldsymbol{u}, p) \in K \times Q$ such that

$$\int_\Omega \alpha |\mathbf{D}\boldsymbol{u}|^{r-2}\mathbf{D}\boldsymbol{u} : \mathbf{D}(\boldsymbol{v}-\boldsymbol{u})\,\mathrm{d}x - \int_\Omega p\nabla\cdot(\boldsymbol{v}-\boldsymbol{u})\,\mathrm{d}x \geq f(\boldsymbol{v}-\boldsymbol{u}) \quad \forall \boldsymbol{v} \in K$$

$$\int_\Omega q(\nabla\cdot\boldsymbol{u})\,\mathrm{d}x = 0 \quad \forall q \in Q$$

where $K = \{\boldsymbol{v} \in V : \boldsymbol{v}\cdot\boldsymbol{n} \leq 0 \quad \text{on } \Gamma_a\}$

Numerical considerations

The weak formulation can be written as an equivalent **minimization problem**:

Find $\boldsymbol{u} \in \overset{\circ}{K}$ that minimises

$$\mathcal{J}(\boldsymbol{v}) = \frac{1}{r}\int_\Omega \alpha|\mathbf{D}\boldsymbol{v}|^r \,\mathrm{d}x - f(\boldsymbol{v})$$

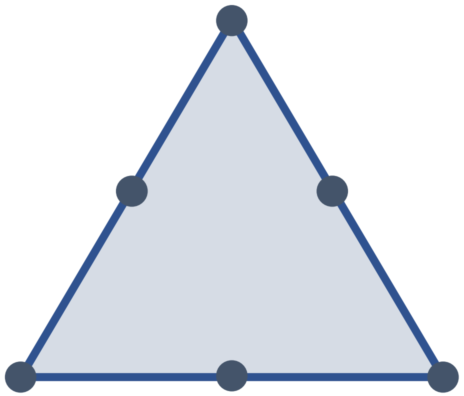
where $\overset{\circ}{K} = \{\boldsymbol{v} \in V : \boldsymbol{v}\cdot\boldsymbol{n} \leq 0 \quad \text{on } \Gamma_a, \quad \nabla\cdot\boldsymbol{v} = 0\}$

Discretization:

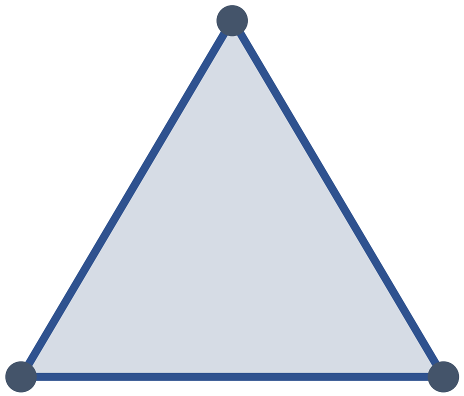
- Velocity and pressure spaces?
- Formulation of the inequality constraint $\boldsymbol{v}\cdot\boldsymbol{n} \leq 0 \quad \text{on } \Gamma_a$ at the discrete level?

Numerical considerations

Discretization of the variational inequality with Taylor-Hood elements, implemented in Firedrake



velocity space $\mathrm{V_h} = \mathbf{CG}_2$
(continuous piecewise quadratic vector polynomials)



pressure space $\mathrm{Q_h} = \mathrm{CG}_1$
(continuous piecewise linear polynomials)

Numerical considerations

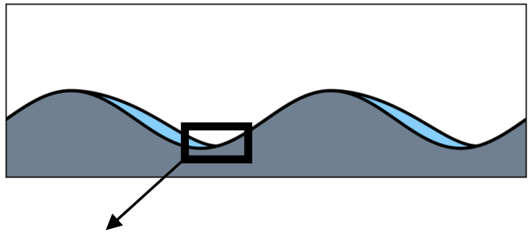
We formulate the inequality constraint in terms of a **discrete normal trace operator**

$$\gamma_h : V_h \to \Sigma_h$$

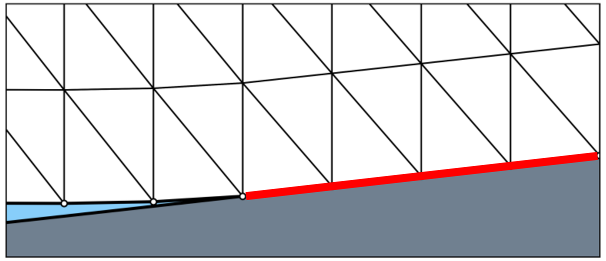
Find $\boldsymbol{u}_h \in \mathring{K}_h$ that minimises

$$\mathcal{J}(\boldsymbol{v}_h) = \frac{1}{r}\int_\Omega \alpha |\mathbf{D}\boldsymbol{v}_h|^r \,\mathrm{d}x - f(\boldsymbol{v}_h)$$

where $\mathring{K}_h = \{\boldsymbol{v}_h \in V_h : \boxed{\gamma_h \boldsymbol{v}_h \leq 0 \quad \text{on } \Gamma_a,} \quad \nabla_h \cdot \boldsymbol{v}_h = 0\}$

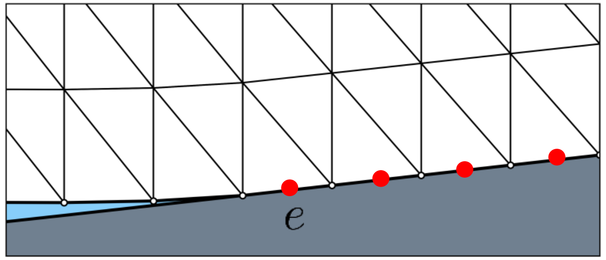


$$\gamma_h \boldsymbol{v}_h(x) = (\boldsymbol{v}_h \cdot \boldsymbol{n})(x)$$



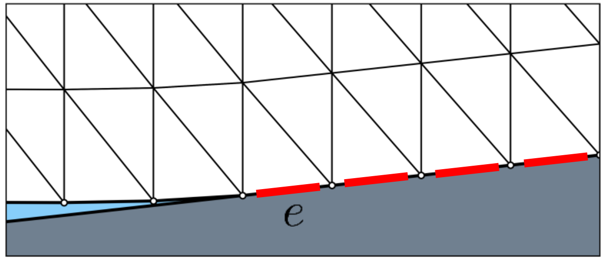
on every point of the attached region

$$\gamma_h \boldsymbol{v}_h|_e = (\boldsymbol{v}_h \cdot \boldsymbol{n})(x_m)$$



on midpoints x_m of edges e

$$\gamma_h \boldsymbol{v}_h|_e = \frac{1}{|e|}\int_e \boldsymbol{v}_h \cdot \boldsymbol{n}$$



average-wise on edges

Numerical considerations

Most natural approach: **enforce the inequality constraint on every point of the attached region:**

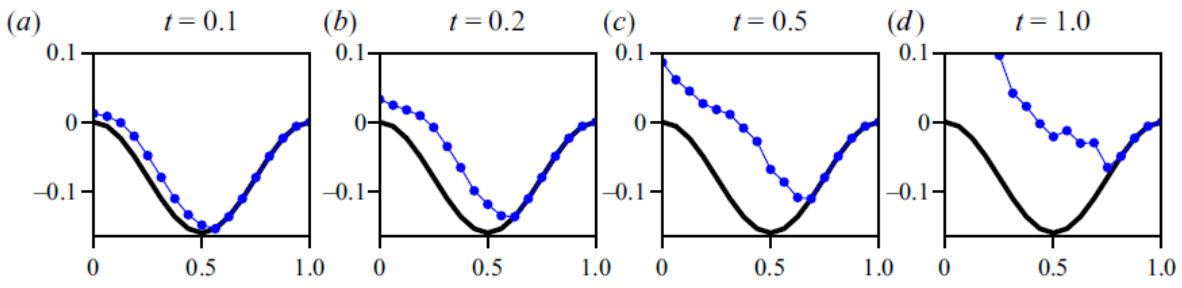
Find $\boldsymbol{u}_h \in \mathring{K}_h$ that minimises

$$\mathcal{J}(\boldsymbol{v}_h) = \frac{1}{r}\int_\Omega \alpha |\mathbf{D}\boldsymbol{v}_h|^r \, \mathrm{d}x - f(\boldsymbol{v}_h)$$

where $\mathring{K}_h = \{\boldsymbol{v}_h \in V_h : \boldsymbol{v}_h \cdot \boldsymbol{n} \leq 0 \quad \text{on } \Gamma_a, \quad \nabla_h \cdot \boldsymbol{v}_h = 0\}$

We can solve this system with e.g. a **penalty method**.

Our experience: inaccurate evolution of the free boundary!



Evolution of cavity with penalty method

Numerical considerations

We obtain much better numerical results by enforcing an **edge-wise average discrete inequality constraint**

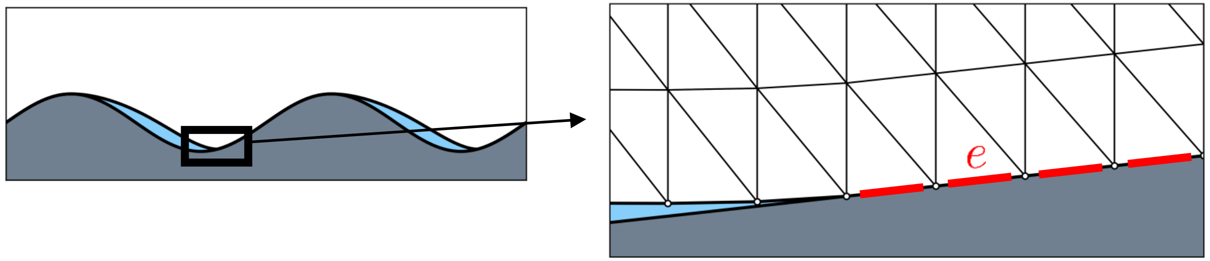
Find $\boldsymbol{u}_h \in \mathring{K}_h$ that minimises

$$\mathcal{J}(\boldsymbol{v}_h) = \frac{1}{r}\int_\Omega \alpha |\mathbf{D}\boldsymbol{v}_h|^r \,\mathrm{d}x - f(\boldsymbol{v}_h)$$

where $\mathring{K}_h = \left\{\boldsymbol{v}_h \in V_h : \frac{1}{e}\int_e \boldsymbol{v}_h \cdot \boldsymbol{n} \leq 0 \quad \forall \text{ edges } e \text{ on } \Gamma_a, \quad \nabla_h \cdot \boldsymbol{v}_h = 0\right\}$

We solve this system numerically by enforcing the inequality constraint with a **Lagrange multiplier**.

$$\gamma_h \boldsymbol{v}_h|_e = \frac{1}{|e|}\int_e \boldsymbol{v}_h \cdot \boldsymbol{n}$$



average-wise on edges

A numerical algorithm for subglacial cavitation

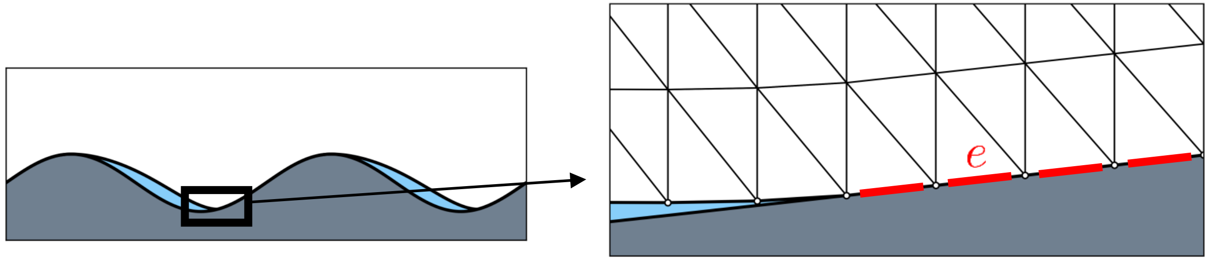
Discrete Stokes variational inequality with a Lagrange multiplier:

Find $(\boldsymbol{u}_h, p_h, \lambda_h) \in V_h \times Q_h \times \mathrm{DG}_0(\Gamma_a)$ such that

$$\int_\Omega \alpha |\mathbf{D}\boldsymbol{u}_h|^{r-2} \mathbf{D}\boldsymbol{u}_h : \mathbf{D}\boldsymbol{v}_h \, \mathrm{d}x - \int_\Omega p_h (\nabla \cdot \boldsymbol{v}_h) \, \mathrm{d}x - \int_{\Gamma_a} \lambda_h (\boldsymbol{v}_h \cdot \boldsymbol{n}) \, \mathrm{d}s = f(\boldsymbol{v}_h) \quad \forall \boldsymbol{v} \in V_h$$

$$\int_\Omega q_h (\nabla \cdot \boldsymbol{u}_h) \, \mathrm{d}x = 0 \quad \forall q_h \in Q_h$$

$$\frac{1}{e} \int_e \boldsymbol{u}_h \cdot \boldsymbol{n} \leq 0, \quad \lambda_h|_e \leq 0, \quad \lambda_h|_e \left(\frac{1}{e} \int_e \boldsymbol{u}_h \cdot \boldsymbol{n} \leq 0 \right) = 0 \quad \forall \text{ edges } e \text{ on } \Gamma_a$$



A numerical algorithm for subglacial cavitation

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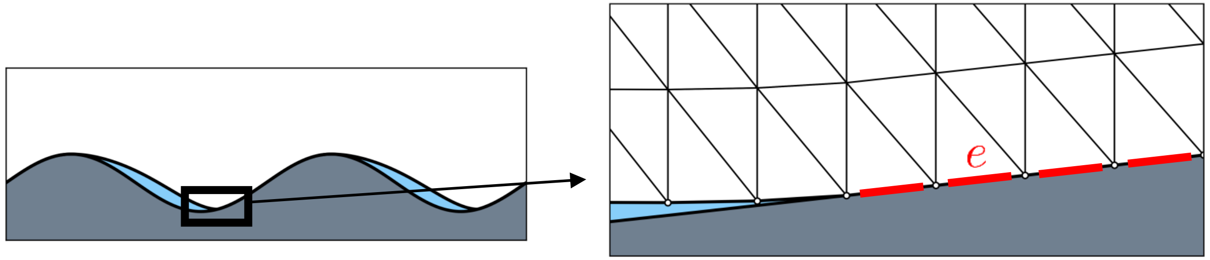
$$\int_\Omega q_h (\nabla \cdot \boldsymbol{u}_h) \,\mathrm{d}x = 0 \quad \forall q_h \in Q_h$$

$$\int_{\Gamma_a} [-\lambda_h + \langle \boldsymbol{u}_h \cdot \boldsymbol{n} \rangle]_+ \, \mu_h \,\mathrm{d}s = 0 \quad \forall \mu_h \in \mathrm{DG}_0(\Gamma_a)$$

$$[f]_+ = \max\{0, f\}$$

$$\langle f \rangle_e = \frac{1}{e} \int_e f$$



A numerical algorithm for subglacial cavitation

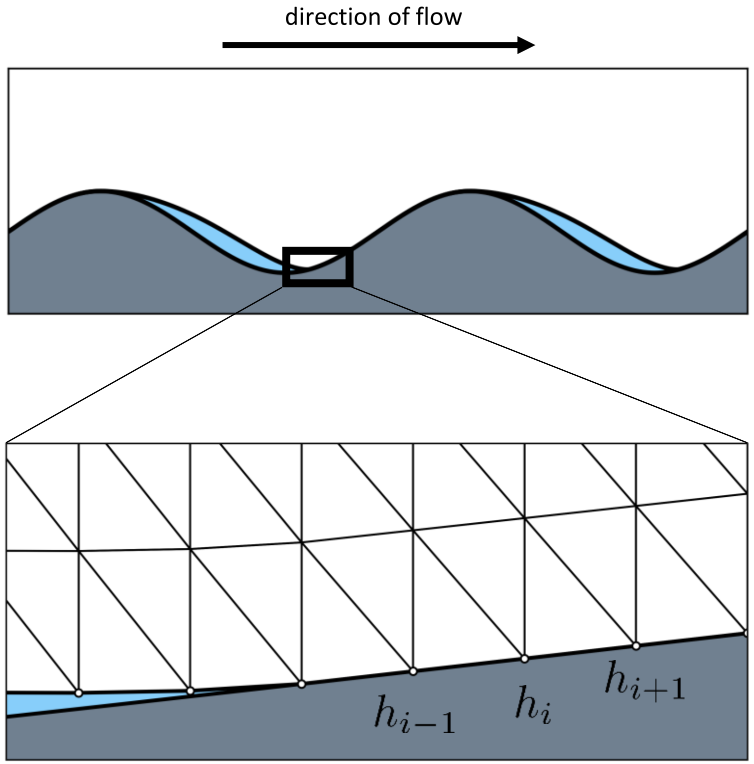
Discrete free boundary equation

We consider a forward Euler FD discretization

$$\partial_t h = -\left[1 + (\partial_x h)^2\right]^{-1/2} \boldsymbol{u} \cdot \boldsymbol{n}$$

$$\downarrow$$

$$\frac{h_i^{k+1} - h_i^k}{\Delta t} = -\left[\left(\partial_x h^k\right)_i^2 + 1\right]^{-1/2} [\boldsymbol{u}_h^k \cdot \boldsymbol{n}]_i.$$



Two considerations:

- Choice of $[\boldsymbol{u}_h^k \cdot \boldsymbol{n}]_i$ should be **compatible** with our choice of **discrete normal trace operator**.
- **Stabilisation** due to the advective nature of the problem

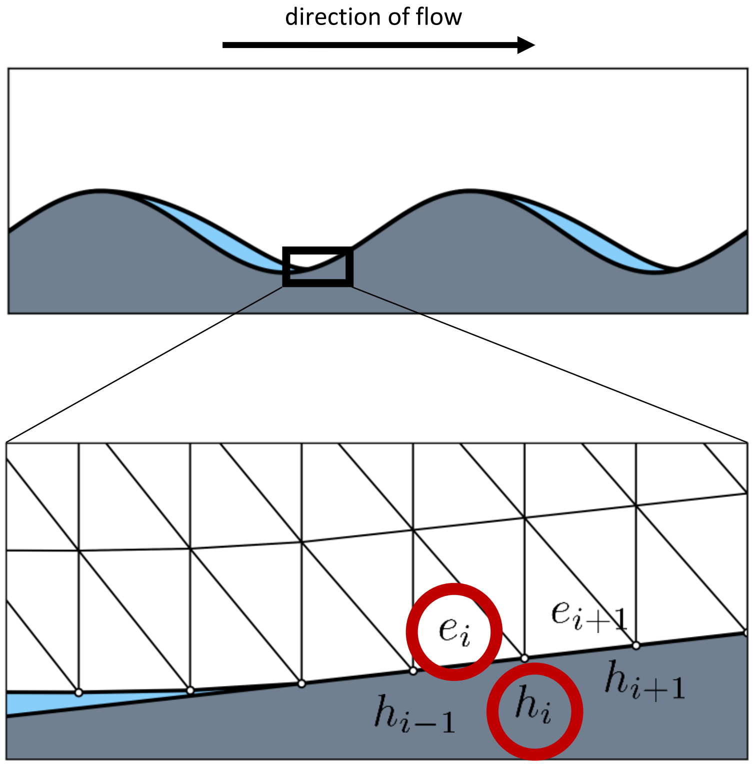
$$\begin{aligned}\partial_t h &= -\left[1 + (\partial_x h)^2\right]^{-1/2} \boldsymbol{u} \cdot \boldsymbol{n} \\ &= -u\,\partial_x h + v.\end{aligned}$$

A numerical algorithm for subglacial cavitation

Discrete free boundary equation

Our discrete free boundary formulation:

$$\frac{h_i^{k+1} - h_i^k}{\Delta t} = -\left[\left(\partial_x h^k\right)_i^2 + 1\right]^{-1/2} \left[\frac{1}{|e_i|}\int_{e_i} \boldsymbol{u}_h^k \cdot \boldsymbol{n}\right].$$



Remarks:

- We set

$$[\boldsymbol{u}_h^k \cdot \boldsymbol{n}]_i = \frac{1}{|e_i|}\int_{e_i} \boldsymbol{u}_h^k \cdot \boldsymbol{n} = \gamma_h \boldsymbol{u}_h^k|_{e_i}$$

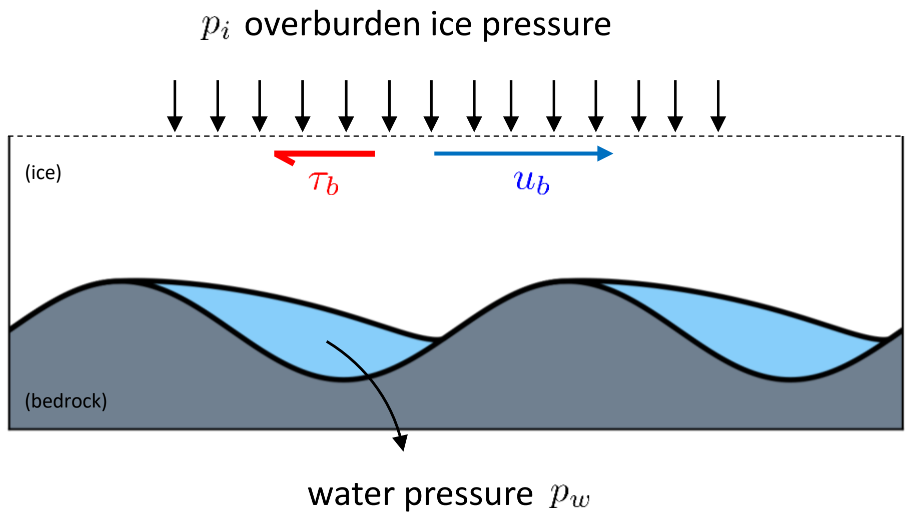
because we enforce a piece-wise constant version of the contact boundary conditions exactly.

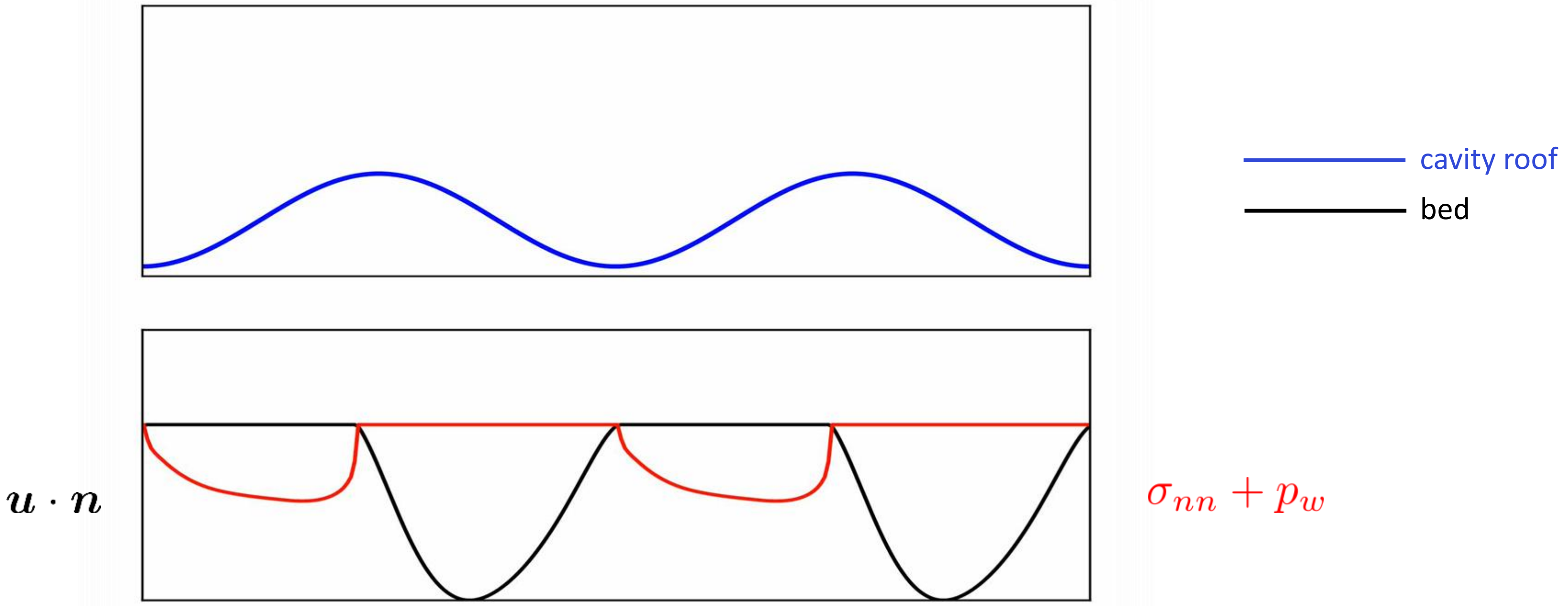
- For each node, we take the edge **upstream** to implement a form of **upwinding** that stabilizes the problem!

Numerical results

Computation of steady states –

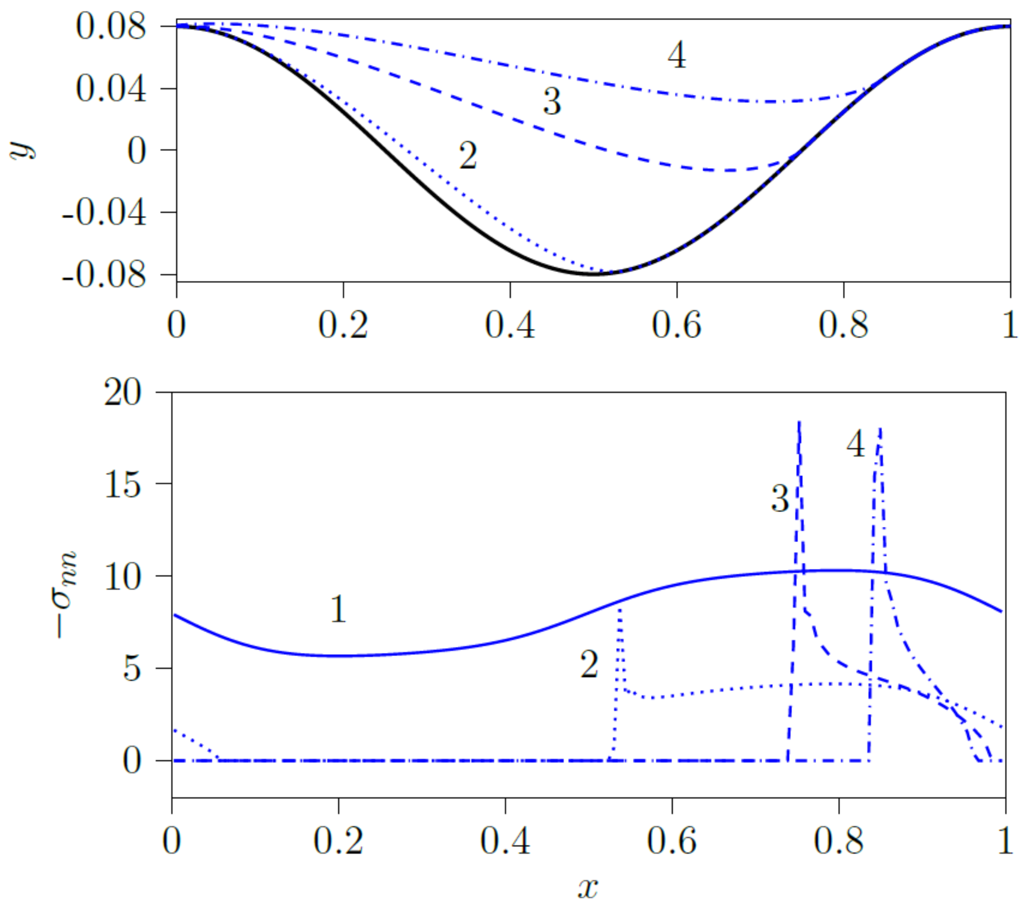
Fix u_b, p_i and p_w - evolve in time to steady cavity



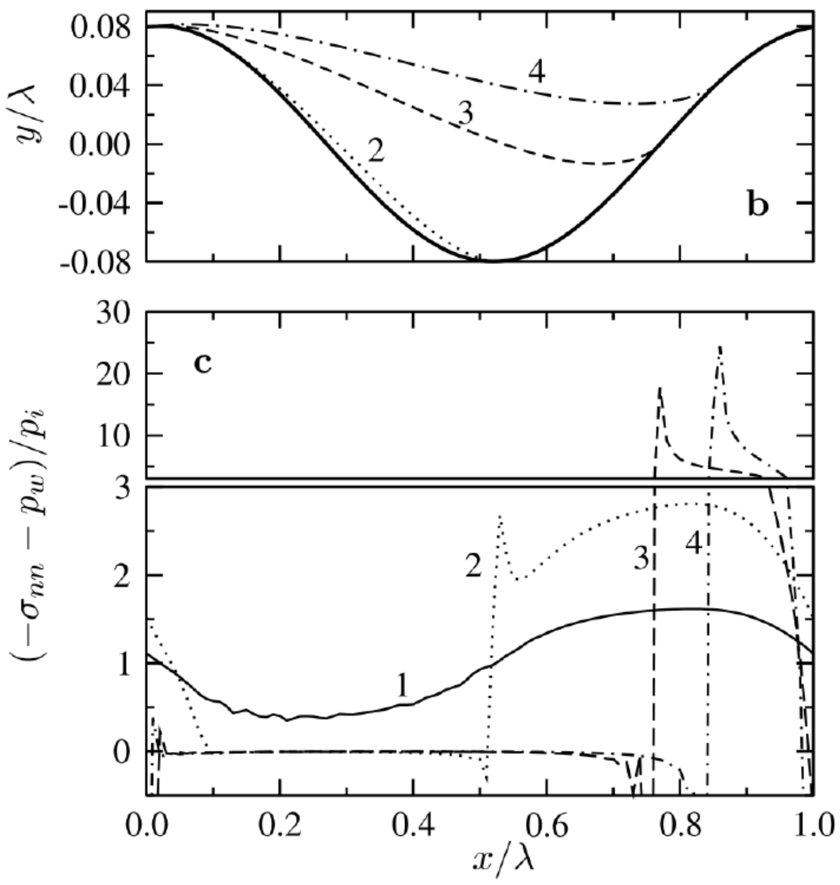


Numerical results

Our method



Elmer/Ice (Gagliardini et al. 2007)



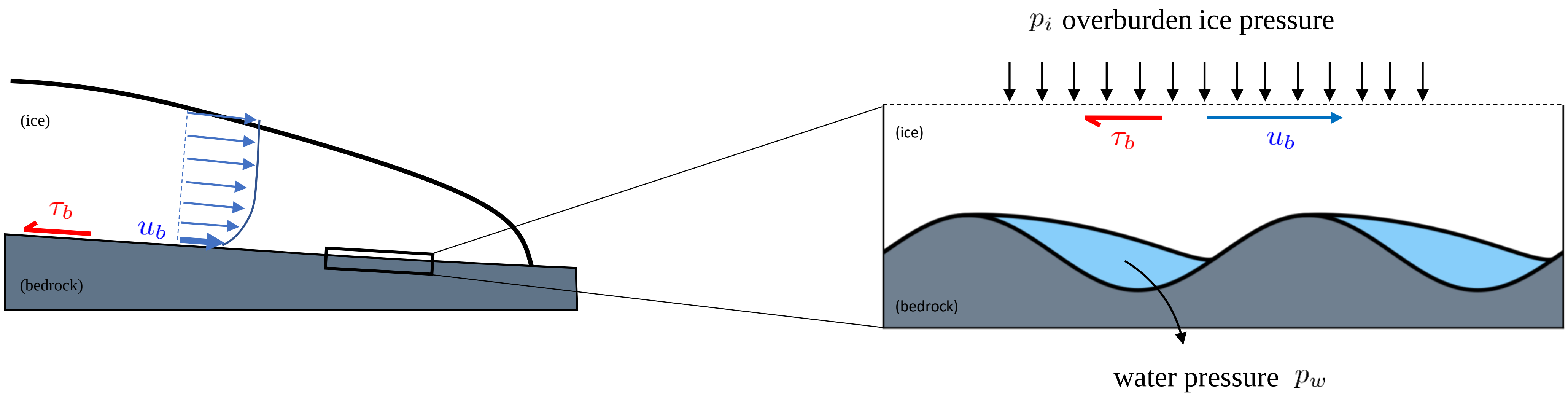
Numerical results

As a numerical example, we compute a **friction law**

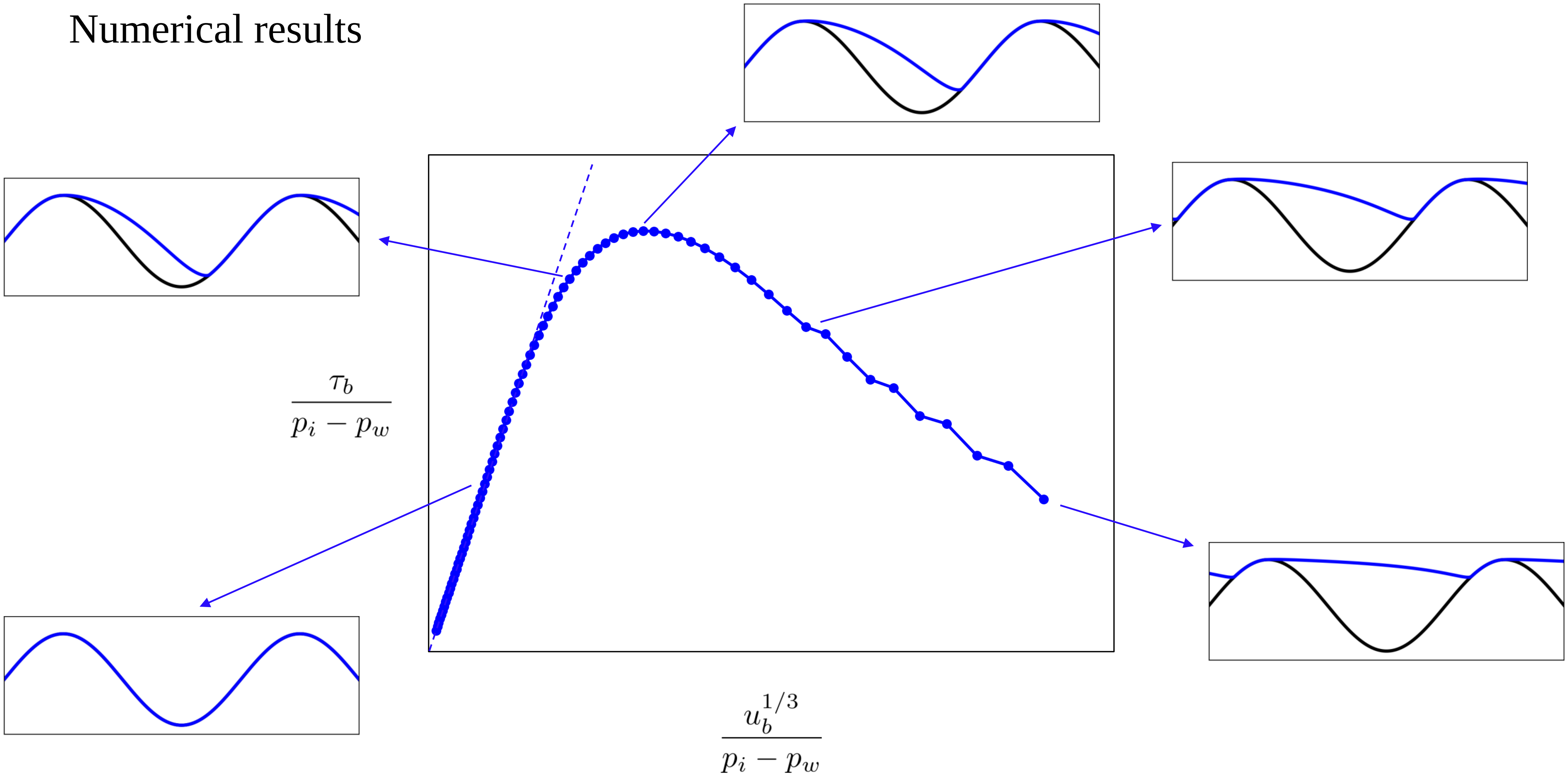
$$\tau_b = f(u_b, p_i - p_w)$$

over a sinusoidal bedrock by finding **steady states** for different pairs $(u_b, p_i - p_w)$ and computing

$$\tau_b = \int_{\Gamma_b} (\sigma_{nn} + p_w)\, n_x \, \mathrm{d}s.$$



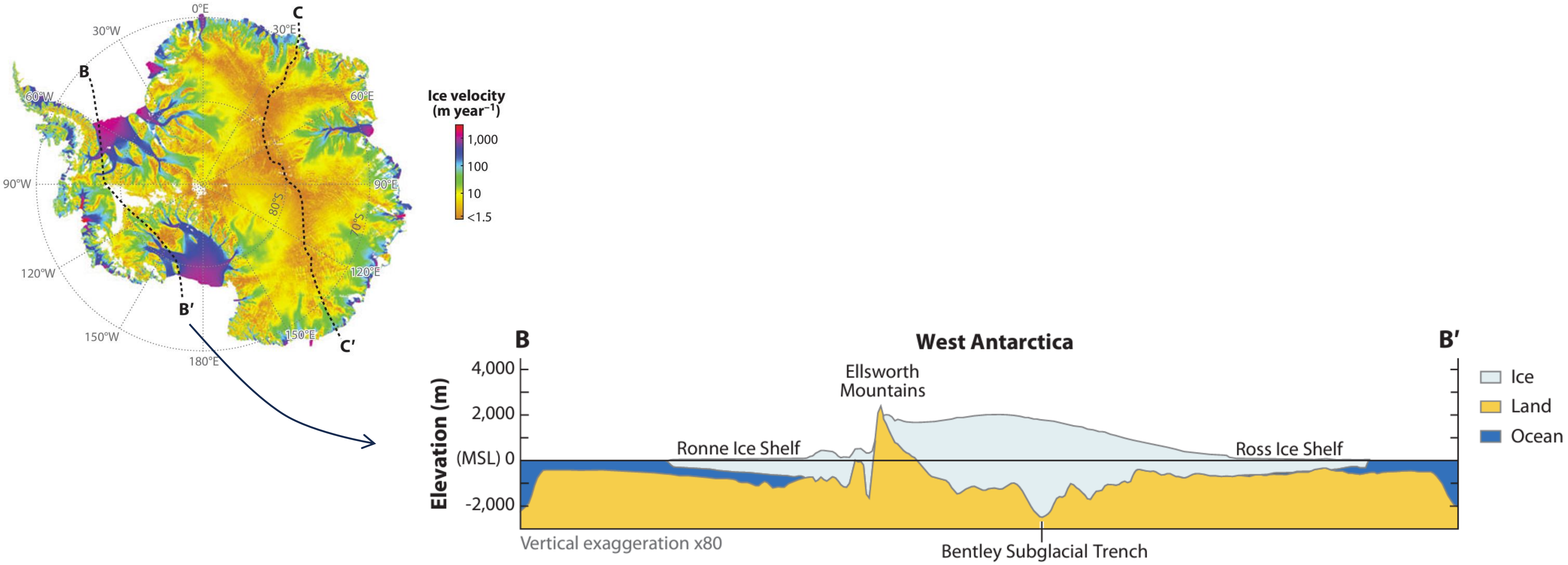
Numerical results



Numerical computation of steady grounding line configurations

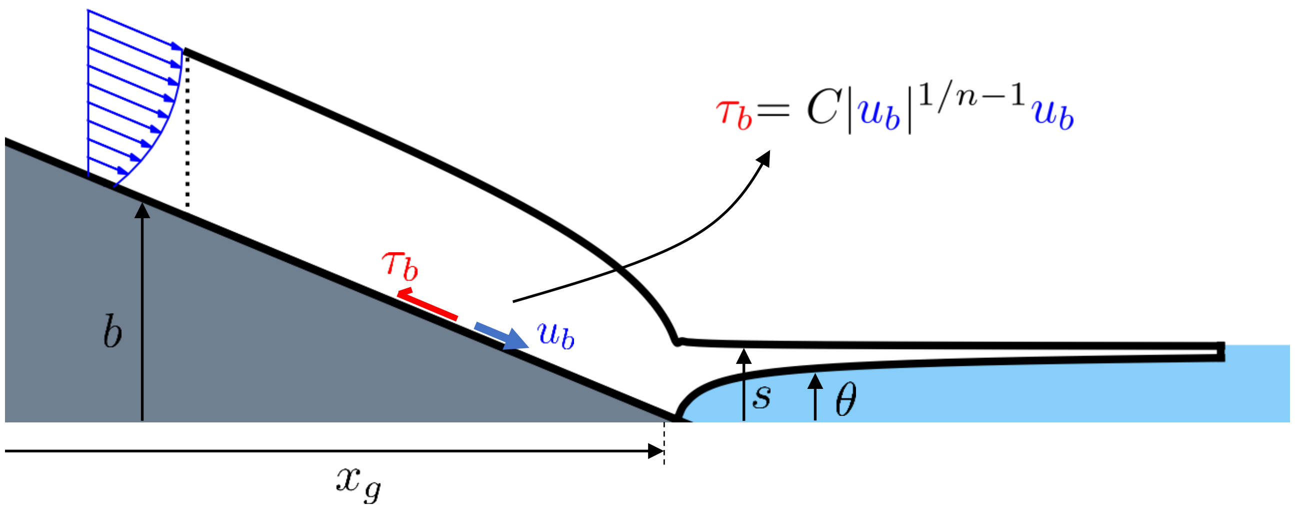
Marine ice sheets flow from the continent and into the ocean, where the go afloat the **grounding line**.

Example – the **Antarctic ice sheet**:



Numerical computation of steady grounding line configurations

Our setup - the parallel slab marine ice sheet:

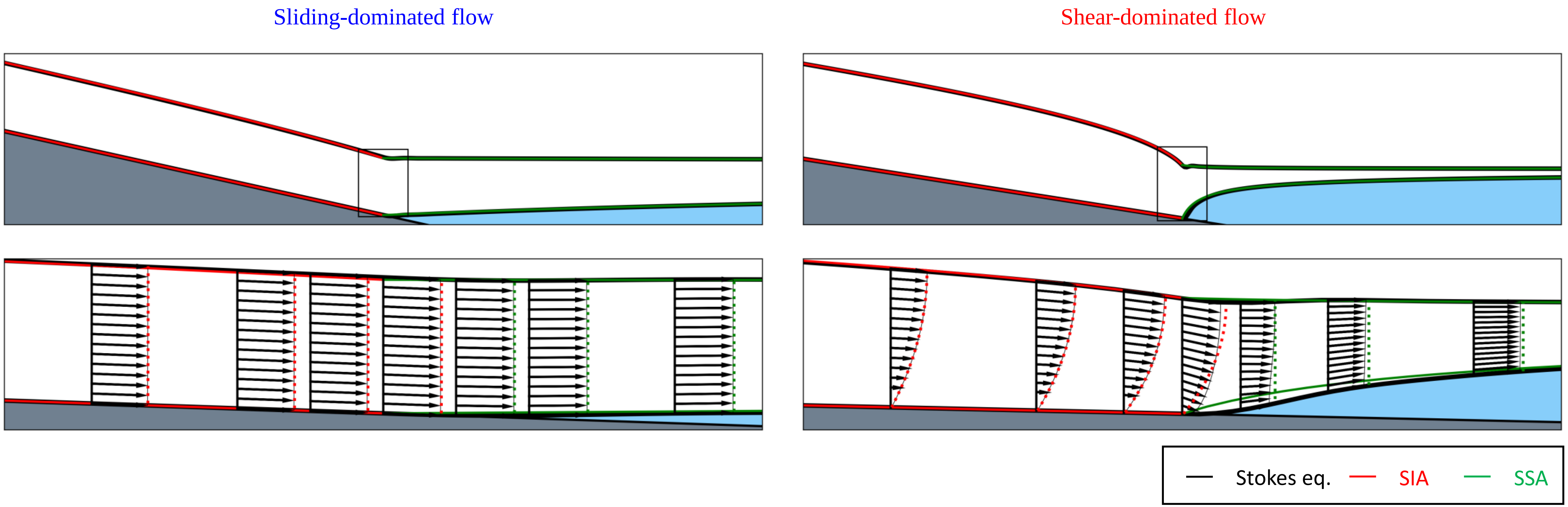


Objective: Explore **steady flux-thickness relationship** $q(x_g) = \int_\theta^s u\,\mathrm{d}z \longleftrightarrow h(x_g)$ for different flow regimes.

- Effect of **bedrock geometry**?
- **Single or multi-valued function**?

Numerical computation of steady grounding line configurations

Derive approximate flux-thickness relationship with **depth-integrated models**



Conclusions

Some comments.

- Accurate computations of viscous contact problems are achieved by **enforcing a discrete piecewise constant contact boundary condition exactly** and **evolving the free boundary with edge-averaged normal velocities** in a stable manner.

- We evolve the system in time to find steady states – this is time consuming and, at times, not robust.

Future improvements?

Solve the variational inequality and the free boundary simultaneously:

- more stable time-stepping schemes,

- possibility of **solving directly for steady states** with e.g. Newton's method.

Thank you very much for your attention!

Elephant Foot Glacier, NE Greenland (Copernicus Sentinel-2A data, August 2016)