### DPG Fundamentals

#### UW formulation

\[ \begin{align*}
    \{ & u \in D(A) \} \quad \Rightarrow \quad \{ u \in L^2(\Omega) \\
    A u = f \quad \Rightarrow \quad (u, A^* v) = (f, v) \quad v \in D(A^*) \quad \Rightarrow \quad \{ u \in L^2(\Omega), \; \hat{u} \in \hat{U} \\
    (u, A^* v) + \langle \hat{u}, v \rangle = (f, v) \quad v \in H_{A^*}(\Omega_h) \}
\end{align*} \]

#### Inf-sup constant $\gamma$ depends upon boundedness below constant $\alpha$ and scaling parameter $\beta$ in the adjoint graph norm

\[
\alpha \| u \| \leq \| Au \|, \; u \in D(A) \\
\| v \|_V^2 := \| A^* v \|^2 + \beta^2 \| v \|^2 \quad \Rightarrow \quad \gamma \geq [1 + \left( \frac{\beta}{\alpha} \right)^2]^{-1/2}.
\]

#### (Ideal) DPG reproduces the stability of the continuous problem

\[
\| u - u_h \|^2 \leq \left[ 1 + \left( \frac{\beta}{\alpha} \right)^2 \right] \{ \inf_{w_h \in U_h} \| u - w_h \|^2 + \inf_{\hat{w}_h \in \hat{U}_h} \| \hat{u} - \hat{w}_h \|^2 \}
\]
Full Envelope UW Formulation for Linear Waveguide Problem (1)

**Def.** Full envelope operator

\[ \tilde{A}\tilde{u} := e^{ikz}A(e^{-ikz}\tilde{u}) \]

**Thm.** Full envelope operator inherits boundedness below constant from the original operator

\[ \|Au\| \geq \alpha \|u\| \iff \|\tilde{A}\tilde{u}\| \geq \alpha \|	ilde{u}\| \]

Proof:

\[ \|\tilde{A}\tilde{u}\| = \|e^{ikz}A(e^{-ikz}\tilde{u})\| = \|A(e^{-ikz}\tilde{u})\| \geq \alpha \|e^{-ikz}\tilde{u}\| = \alpha \|	ilde{u}\| \]

**Thm.** The boundedness below constant depends inversely linearly upon waveguide length \( L \) (the subject of this talk)

\[ \|Au\| \geq \frac{\alpha_0}{L} \|u\| \]

\[ =: \alpha \]

---

Positive Effect of Small $\beta$ on Pollution

Pollution error in a 3D rectangular waveguide for ultraweak DPG Maxwell with test norm:

$$\| u \|_{V(\Omega_h)}^2 = \| \text{curl} F - i\omega \varepsilon G \|^2 + \| \text{curl} G + i\omega F \|^2 + \beta^2 \left( \| F \|^2 + \| G \|^2 \right).$$
ANALYSIS OF A NON-HOMOGENEOUS EM WAVEGUIDE PROBLEM(2)

L. Demkowicz\textsuperscript{a}, M. Melenk\textsuperscript{b}, S. Henneking\textsuperscript{a} and J. Badger\textsuperscript{a}

\textsuperscript{a}Oden Institute, The University of Texas at Austin
\textsuperscript{b}Technical University of Vienna

Sponsored by Air Force Grant # FA9550-19-1-0237

FEMLLNL Seminar Series
Livermore, Apr 25, 2023

Outline

1. Eigensystems
2. Homogeneous Waveguide
3. Cylindrical Waveguide
4. Perturbation of Self-Adjoint Operators
5. Decoupling the Equations
6. Estimation
Eigensystems

Let \( E = (E_1, E_2, E_3) \), \( e_z = (0, 0, 1) \). We will use the 2D identities:

\[
\begin{align*}
e_z \times (e_z \times E_t) &= -E_t \\
e_z \times (\nabla \times E_3) &= \nabla E_3 \\
curl(e_z \times E_t) &= \text{div} E_t \\
ev_z \times \nabla E_3 &= -\nabla \times E_3 \\
div(e_z \times E_t) &= -\text{curl} E_t.
\end{align*}
\]

The original system of equations,

\[
\begin{align*}
\nabla \times E - i\omega H &= f \\
\nabla \times H + i\omega \epsilon E &= g
\end{align*}
\]

translates into:

\[
\begin{align*}
\nabla \times E_3 + e_z \times \frac{\partial}{\partial z} E_t - i\omega H_t &= f_t \\
curl E_t - i\omega H_3 &= f_3 \\
\nabla \times H_3 + e_z \times \frac{\partial}{\partial z} H_t + i\omega \epsilon E_t &= g_t \\
curl H_t + i\omega \epsilon E_3 &= g_3.
\end{align*}
\]
Let $E = (E_1, E_2, E_3)$, $e_z = (0, 0, 1)$. We will use the 2D identities:

\[ e_z \times (e_z \times E) = -E_t \]
\[ e_z \times (\nabla \times E_3) = \nabla E_3 \quad e_z \times \nabla E_3 = -\nabla \times E_3 \]
\[ \text{curl}(e_z \times E_t) = \text{div} E_t \quad \text{div}(e_z \times E_t) = -\text{curl} E_t. \]

The original system of equations,

\[ \nabla \times E - i\omega H = f \quad \nabla \times H + i\omega \epsilon E = g \]

Multiplying the first and third equations by $i\omega e_z \times$, we obtain:

\[ \begin{align*}
\nabla i\omega E_3 - \frac{\partial}{\partial z} i\omega E_t + \omega^2 e_z \times H_t &= i\omega e_z \times f_t \\
\text{curl} E_t - i\omega H_3 &= f_3 \\
\nabla i\omega H_3 - \frac{\partial}{\partial z} i\omega H_t - \omega^2 e_z \times \epsilon E_t &= i\omega e_z \times g_t \\
\text{curl} H_t + i\omega \epsilon E_3 &= g_3.
\end{align*} \]
Waveguide Problem and Its Adjoint

The eigensystem corresponding to the first order system operator, and $e^{i\beta z}$ ansatz in $z$:

\[
\begin{align*}
E_t & \in H_0(\text{curl}, D), \ E_3 \in H_0^1(D) \\
H_t & \in H(\text{curl}, D), \ H_3 \in H^1(D) \\
\begin{align*}
i\omega \nabla E_3 + \omega^2 e_z \times H_t &= -\omega \beta E_t \\
\text{curl } E_t - i\omega H_3 &= 0 \\
i\omega \nabla H_3 - \omega^2 e_z \times \epsilon E_3 &= -\omega \beta H_t \\
\text{curl } H_t + i\omega \epsilon E_3 &= 0.
\end{align*}
\end{align*}
\] (1.2)

The system corresponding to the adjoint:

\[
\begin{align*}
F_t & \in H(\text{div}, D), \ F_3 \in H^1(D) \\
G_t & \in H_0(\text{div}, D), \ G_3 \in H_0^1(D) \\
\begin{align*}
\nabla \times F_3 + \omega^2 e_z \times \epsilon G_t &= -\omega \beta F_t \\
i\omega (\text{div } F_t - \epsilon G_3) &= 0 \\
\nabla \times G_3 - \omega^2 e_z \times F_t &= -\omega \beta G_t \\
i\omega (\text{div } G_t + F_3) &= 0.
\end{align*}
\end{align*}
\] (1.3)
Eigensystems

Eliminating $E_3$ and $H_3$ from system (1.2), we obtain a simplified but second order “EH system” for $E_t, H_t$ only.

\[
\begin{aligned}
E_t &\in H_0(\text{curl}, D), \text{curl } E_t \in H^1(D) \\
H_t &\in H(\text{curl}, D), \frac{1}{\epsilon} \text{curl } H_t \in H^1_0(D) \\
-\nabla (\frac{1}{\epsilon} \text{curl } H_t) + \omega^2 e_z \times H_t &= -\omega \beta E_t \\
\nabla(\text{curl } E_t) - \omega^2 e_z \times \epsilon E_t &= -\omega \beta H_t.
\end{aligned}
\]

(1.4)

Similarly, eliminating $F_3$ and $G_3$ from system (1.3), we obtain a second order “FG system” for $F_t, G_t$ only.

\[
\begin{aligned}
F_t &\in H(\text{div}, D), \frac{1}{\epsilon} \text{div } F_t \in H^1_0(D) \\
G_t &\in H_0(\text{div}, D), \text{div } G_t \in H^1(D) \\
-\nabla \times \text{div } G_t + \omega^2 e_z \times \epsilon G_t &= -\omega \beta F_t \\
\nabla \times (\frac{1}{\epsilon} \text{div } F_t) - \omega^2 e_z \times F_t &= -\omega \beta G_t.
\end{aligned}
\]

(1.5)

One can check that the operator in (1.5) corresponds to the adjoint of operator in (1.4). Notice how the BCs on $E_3, G_3$ have been inherited by curl $H_t$ and div $F_t$. 
Reduction to single variable eigensystems

Assume $\beta \neq 0$. Solving (1.4) for $H_t$,

$$H_t = -\frac{1}{\omega \beta}[\nabla \text{curl} E_t - \omega^2 e_z \times \epsilon E_t]$$

$$\text{curl} H_t = \frac{\omega}{\beta} \text{curl}(e_z \times \epsilon E_t) = \frac{\omega}{\beta} \text{div} \epsilon E_t$$

and substituting it into (1.4), we obtain an "E eigenvalue problem" for $E_t$ alone.

$$E_t \in H_0(\text{curl}, D), \text{curl} E_t \in H^1(D), \frac{1}{\epsilon} \text{div} \epsilon E_t \in H_0^1(D)$$

$$\nabla \times \text{curl} E_t - \omega^2 \epsilon E_t - \nabla \left( \frac{1}{\epsilon} \text{div} \epsilon E_t \right) = -\beta^2 E_t.$$ (1.7)

Similarly, solving (1.4) for $E_t$,

$$E_t = -\frac{1}{\omega \beta}[-\nabla \left( \frac{1}{\epsilon} \text{curl} H_t \right) + \omega^2 e_z \times H_t]$$

$$\text{curl} E_t = -\frac{\omega}{\beta} \text{curl}(e_z \times H_t) = -\frac{\omega}{\beta} \text{div} H_t$$

and substituting it into (1.4), we obtain an "H eigenvalue problem" for $H_t$ alone.

$$H_t \in H(\text{curl}, D) \cap H_0(\text{div}, D), \frac{1}{\epsilon} \text{curl} H_t \in H_0^1(D), \text{div} H_t \in H^1(D)$$

$$\epsilon \nabla \times \left( \frac{1}{\epsilon} \text{curl} H_t \right) - \omega^2 \epsilon H_t - \nabla \left( \text{div} H_t \right) = -\beta^2 H_t.$$ (1.9)

Note that BC: $n \times E_t = 0$ implies BC: $n \cdot H_t = 0$. 
Same for the Adjoint Problem

Solving (1.5) for $G_t$,

\[
G_t = -\frac{1}{\omega \beta} [\nabla \times (\frac{1}{\epsilon} \text{div} F_t) - \omega^2 e_z \times F_t]
\]

and substituting it into (1.5) for $F_t$ alone.

\[
\begin{align*}
F_t & \in H_0(\text{curl}, D) \cap H(\text{div}, D), \quad \frac{1}{\epsilon} \text{div} F_t \in H^1_0(D), \quad \text{curl} F_t \in H^1(D) \\
\nabla \times \text{curl} F_t - \omega^2 \text{F}_t - \epsilon \nabla (\frac{1}{\epsilon} \text{div} F_t) &= -\gamma^2 F_t \\
\end{align*}
\]

(1.10)

Note that BC: $n \cdot G_t = 0$ implies BC: $n \times F_t = 0$. Similarly, solving (1.5) for $F_t$,

\[
F_t = -\frac{1}{\omega \beta} [\nabla \times \text{div} G_t] + \omega^2 e_z \times \epsilon G_t
\]

\[
\text{div} F_t = -\frac{\omega}{\beta} \text{div} (e_z \times \epsilon G_t) = -\frac{\omega}{\beta} \text{curl} \epsilon G_t
\]

(1.11)

and substituting it into (1.5) for $G_t$ alone.

\[
\begin{align*}
G_t & \in H_0(\text{div}, D), \quad \text{div} G_t \in H^1(D), \quad \frac{1}{\epsilon} \text{curl} \epsilon G_t \in H^1_0(D) \\
\nabla \times (\frac{1}{\epsilon} \text{curl} \epsilon G_t) - \omega^2 \epsilon G_t - \nabla (\text{div} G_t) &= -\gamma^2 G_t \\
\end{align*}
\]

(1.12)

(1.13)
Lemma (1)

(a) Let \((E_t, H_t, -\omega \beta)\) be an eigenpair for EH system (1.4). Then \((E_t, -\beta^2)\) solves the E problem (1.7), and \((H_t, -\beta^2)\) solves the H problem (1.9).

(b) Conversely, if \((E_t, -\beta^2)\) is an eigenpair for the E problem (1.7), and we define \(H_t\) by

\[
H_t = \frac{1}{\omega(\pm \beta)} \left( -\nabla \text{curl} E_t + \omega^2 e_z \times \epsilon E_t \right)
\]

then \((E_t, H_t, -\omega(\pm \beta))\) is an eigenpair for EH system (1.4). Each eigenpair for E problem (1.7) generates two eigenpairs for EH problem (1.4).

(c) Similarly, if \((H_t, -\beta^2)\) is an eigenpair for H problem (1.9), and we define \(E_t\) by:

\[
E_t = \frac{1}{\omega(\pm \beta)} \left( \nabla \left( \frac{1}{\epsilon} \text{curl} H_t \right) - \omega^2 e_z \times H_t \right)
\]

then \((E_t, H_t, -\omega(\pm \beta))\) is an eigenpair for EH system (1.4). Each eigenpair for H problem (1.9) generates two eigenpairs for EH problem (1.4).

In particular, Lemma 1 implies that \(E\) and \(H\) eigenproblems have the same eigenvalues \(\beta^2\).
Lemma (2)

(a) Let \((F_t, G_t, \omega \gamma)\) be an eigenpair for FG system (1.5). Then \((G_t, -\gamma^2)\) solves G problem (1.13) and \((F_t, -\gamma^2)\) solves F problem (1.11).

(b) Conversely, if \((F_t, -\gamma^2)\) is an eigenpair for F problem (1.11), and we define \(G_t\) by

\[
G_t = \frac{1}{\omega(\pm \gamma)} \left(-\nabla \text{curl} F_t + \omega^2 \epsilon_z \times \epsilon F_t\right)
\]

then \(((F_t, G_t), \omega(\pm \gamma))\) is an eigenpair for FG system (1.5). Each eigenpair for F problem (1.11) generates two eigenpairs for FG system (1.5).

(c) Similarly, if \((G_t, -\gamma^2)\) is an eigenpair for G problem (1.13), and we define \(F_t\) by:

\[
F_t = \frac{1}{\omega(\pm \gamma)} \left(\nabla \left(\frac{1}{\epsilon} \text{curl} G_t\right) - \omega^2 \epsilon_z \times G_t\right)
\]

then \(((F_t, G_t), \omega(\pm \gamma))\) is an eigenpair for FG system (1.5). Each eigenpair for G problem (1.13) generates two eigenpairs for FG system (1.5).

In particular, Lemma 2 implies that F and G eigenproblems have the same eigenvalues \(\gamma^2\).
Lemma (3)

\((E_t, -\beta^2)\) is an eigenpair for \(E\) problem (1.7) if and only if \((G_t := e_z \times E_t, -\beta^2)\) is an eigenpair for \(G\) problem (1.13). Similarly, \((H_t, -\beta^2)\) is an eigenpair for \(H\) problem (1.9) if and only if \((F_t := e_z \times H_t, -\beta^2)\) is an eigenpair for \(F\) problem (1.11). In particular, this implies that all four individual eigenproblems share the same eigenvalues.
Outline

1. Eigensystems
2. Homogeneous Waveguide
3. Cylindrical Waveguide
4. Perturbation of Self-Adjoint Operators
5. Decoupling the Equations
6. Estimation
Homogeneous Waveguide

For \( \epsilon = 1 \), the \( E \) problem (1.7) reduces to (\( E = E_t \)):

\[
\begin{align*}
E & \in H_0(\text{curl}, D), \text{ curl } E \in H^1(D), \text{ div } E \in H_0^1(D) \\
\nabla \times \text{ curl } E - \omega^2 E - \nabla(\text{div } E) &= -\beta^2 E. \\
\end{align*}
\]

\( (2.14) \)

We get the same equation for the \( H \) problem (1.9) but with different BCs (\( H = H_t \)):

\[ H \in H(\text{curl}, D) \cap H_0(\text{div}, D), \text{ curl } H \in H_0^1(D), \text{ div } H \in H^1(D). \]

Lemma (4. Helmholtz decompositions)

Let \( D \subset \mathbb{R}^2 \) be a simply connected domain. For every \( E \in L^2(D)^2 \) there exist a unique \( \phi \in H_0^1(D) \) and a unique \( \psi \in H^1(D) \), \( \int_D \psi = 0 \), such that

\[ E = \nabla \phi + \nabla \times \psi. \]

\( (2.15) \)

Similarly, for every \( H \in L^2(D)^2 \) there exist a unique \( \phi \in H_0^1(D) \) and a unique \( \psi \in H^1(D), \int_D \psi = 0 \), such that

\[ H = \nabla \times \phi + \nabla \psi. \]

\( (2.16) \)
Consider now the eigenvalue problem (2.14) and Helmholtz decomposition of $E$. Boundary condition $E_t = 0$ implies that $\psi_t = 0$ on $\partial D$. Substituting (2.15) into (2.14), we obtain:

$$\nabla \times (-\Delta \psi + (\beta^2 - \omega^2)\psi) + \nabla(-\Delta \phi + (\beta^2 - \omega^2)\phi) = 0.$$  \hspace{1cm} (2.17)

The equation above represents the Helmholtz decomposition of zero function. Uniqueness of $\phi$ and $\psi$ in the Helmholtz decomposition implies now that $\Phi = \Psi = 0$. Let $(\lambda_i, \phi_i)$ and $(\mu_j, \psi_j)$ be the Dirichlet and Neumann eigenpairs of the Laplacian in domain $D$. Vanishing of $\Phi$ and $\Psi$ implies that there exist $i, j$ such that

$$\phi = \phi_i, \omega^2 - \beta^2 = \lambda_i \quad \text{and} \quad \psi = \psi_j, \omega^2 - \beta^2 = \mu_j.$$  

If the Dirichlet and Neumann eigenvalues are distinct, eigenvector $E$ must reduce to either gradient or curl. This is the case, e.g., for a circular domain $D$. In the case of a common Dirichlet and Neumann eigenvalue, $\lambda_i = \mu_j$, we obtain a multiple eigenvalue $\beta^2 = \omega^2 - \lambda_i = \omega^2 - \mu_j$, with the eigenspace consisting of vectors:

$$E = A\nabla \times \psi_j + B\nabla \phi_i, \quad A, B \in \mathbb{C}.$$
Lemma (5)

Let \( (\lambda_i, \phi_i) \) and \( (\mu_j, \psi_j) \) denote the Dirichlet and Neumann eigenpairs of the Laplacian in domain \( D \). The eigenvalues \( \beta^2_i \) are classified into the following three families.

(a) \( \beta^2 = \omega^2 - \mu_j \) with \( \mu_j \) distinct from all \( \lambda_i \). The corresponding eigenvectors are curls:

\[
E = \nabla \times \psi_j ,
\]

with multiplicity of \( \beta^2 \) equal to the multiplicity of \( \mu_j \).

(a) \( \beta^2 = \omega^2 - \lambda_i \) with \( \lambda_i \) distinct from all \( \mu_j \). The corresponding eigenvectors are gradients:

\[
E = \nabla \phi_i ,
\]

with multiplicity of \( \beta^2 \) equal to the multiplicity of \( \lambda_i \).

(c) \( \beta^2 = \omega^2 - \mu_j = \omega^2 - \lambda_i \) for \( \mu_j = \lambda_i \). The corresponding eigenvectors are linear combinations of curls and gradients:

\[
E = A\nabla \times \psi_j + B\nabla \phi_i , \quad A, B \in \mathbb{C} ,
\]

with multiplicity of \( \beta^2 \) equal to the sum of multiplicities of \( \mu_j \) and \( \lambda_i \).
Lemma (6)

Let \((\lambda_i, \phi_i)\) and \((\mu_j, \psi_j)\) denote the Dirichlet and Neumann eigenpairs of the Laplacian in domain \(D\). The eigenvalues \(\gamma_i^2\) are classified into the following three families.

\((a)\) \(\gamma^2 = \omega^2 - \mu_j\) with \(\mu_j\) distinct from all \(\lambda_i\). The corresponding eigenvectors are gradients:

\[ H = \nabla \psi_j, \]

with multiplicity of \(\beta^2\) equal to the multiplicity of \(\mu_j\).

\((a)\) \(\gamma^2 = \omega^2 - \lambda_i\) with \(\lambda_i\) distinct from all \(\mu_j\). The corresponding eigenvectors are curls:

\[ H = \nabla \times \phi_i, \]

with multiplicity of \(\gamma^2\) equal to the multiplicity of \(\lambda_i\).

\((c)\) \(\gamma^2 = \omega^2 - \mu_j = \omega^2 - \lambda_i\) for \(\mu_j = \lambda_i\). The corresponding eigenvectors are linear combinations of gradients and curls:

\[ H = A\nabla \psi_j + B\nabla \times \phi_i, \quad A, B \in \mathbb{C}, \]

with multiplicity of \(\gamma^2\) equal to the sum of multiplicities of \(\mu_j\) and \(\lambda_i\).
Outline

1. Eigensystems
2. Homogeneous Waveguide
3. Cylindrical Waveguide
4. Perturbation of Self-Adjoint Operators
5. Decoupling the Equations
6. Estimation
Cylindrical Waveguide

Consider the Dirichlet or Neumann Laplace eigenvalue problem in a unit circle,

\[-\Delta u = \lambda u, \quad \lambda = \nu^2.\]

For Dirichlet problem the operator is positive definite, so \(\nu > 0\), for Neumann problem, \(u = \text{const}\) corresponds to zero eigenvalue, all other eigenvalues are positive as well. Rewriting the operator in polar coordinates \(r, \theta\),

\[-\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right) - \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = \nu^2 u.\]

Separating the variables, \(u = R(r)\Theta(\theta)\), we get

\[-\frac{1}{r} (r R')' \Theta - \frac{1}{r^2} R \Theta'' = \nu^2 R \Theta\]

or,

\[\frac{r (r R')'}{R} + \nu^2 r^2 = -\frac{\Theta''}{\Theta} = k^2\]

where \(k^2\) is a real and positive separation constant. We obtain,

\[\Theta = A \cos k \theta + B \sin k \theta\]

and the periodic BCs on \(u\) and, therefore, \(\Theta\), imply that \(k = 0, 1, 2, \ldots\)
Cylindrical Waveguide

This leads to the Bessel equation in $r$,

$$r(rR')' + (\nu^2 r^2 - k^2)R = 0$$

with solution:

$$R = CJ_k(\nu r) + DY_k(\nu r).$$

Finite energy condition eliminates the second term, $D = 0$.

**Dirichlet BC:** $R(1) = 0$ leads to $\nu$ being a root of the Bessel function $J_k(\nu) = 0$. We have a family of roots (and, therefore Dirichlet Laplace eigenvalues $\nu^2$):

$$\nu = \nu_{k,m}, \ k = 0, 1, 2, \ldots, m = 1, 2, \ldots.$$ For $k = 0$, the roots are simple, with corresponding eigenvectors given by:

$$u = J_0(\nu_{0,m} r).$$

For $k > 0$, we have double eigenvectors with eigenspaces given by:

$$u = J_0(\nu_{k,m} r)(A \cos k\theta + B \sin k\theta).$$

**Neumann BC:** The situation is similar except that we are dealing now with the roots of the derivative of Bessel functions: $J'_k(\lambda) = 0, \ \lambda = \lambda_{k,m}, \ k = 0, 1, 2, \ldots, m = 1, 2, \ldots.$
### Cylindrical Waveguide - Cont.

<table>
<thead>
<tr>
<th>(m/k)</th>
<th>(J_0(x))</th>
<th>(J_1(x))</th>
<th>(J_2(x))</th>
<th>(J_3(x))</th>
<th>(J_4(x))</th>
<th>(J_5(x))</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2.4048</td>
<td>3.8317</td>
<td>5.1356</td>
<td>6.3802</td>
<td>7.5883</td>
<td>8.7715</td>
</tr>
<tr>
<td>2</td>
<td>5.5201</td>
<td>7.0156</td>
<td>8.4172</td>
<td>9.7610</td>
<td>11.0647</td>
<td>12.3386</td>
</tr>
</tbody>
</table>

Roots of Bessel functions \(\nu_{k,m}\).

<table>
<thead>
<tr>
<th>(m/k)</th>
<th>(J'_0(x))</th>
<th>(J'_1(x))</th>
<th>(J'_2(x))</th>
<th>(J'_3(x))</th>
<th>(J'_4(x))</th>
<th>(J'_5(x))</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3.8317</td>
<td>1.8412</td>
<td>3.0542</td>
<td>4.2012</td>
<td>5.3175</td>
<td>6.4156</td>
</tr>
<tr>
<td>2</td>
<td>7.0156</td>
<td>5.3314</td>
<td>6.7061</td>
<td>8.0152</td>
<td>9.2824</td>
<td>10.5199</td>
</tr>
</tbody>
</table>

Roots of derivatives of Bessel functions \(\lambda_{k,m}\).
Outline

1. Eigensystems
2. Homogeneous Waveguide
3. Cylindrical Waveguide
4. Perturbation of Self-Adjoint Operators
5. Decoupling the Equations
6. Estimation
Perturbation Analysis

The $E$ problem (1.7):

$$\nabla \times \text{curl} \, E_t - \omega^2 \epsilon E_t - \nabla \left( \frac{1}{\epsilon} \text{div} \, \epsilon E_t \right) = -\beta^2 E_t$$

is not self-adjoint, but it is a perturbation of the self-adjoint homogeneous $E$ problem for $\epsilon = 1$. The homogeneous problem admits two families of eigenvectors:

$$E_i = \nabla \times \psi_i, \quad \beta_i^2 = \omega^2 - \mu_i$$
$$E_j = \nabla \phi_j, \quad \beta_j^2 = \omega^2 - \lambda_j$$

where $(\mu_i, \psi_i)$ and $(\lambda_j, \phi_j)$ are Neumann and Dirichlet eigenpairs for the Laplace operator. Consider now a perturbation,

$$\epsilon = 1 + \delta \epsilon, \quad E := E + \delta E, \quad \beta^2 := \beta^2 + \delta \beta^2.$$ 

Plugging the perturbations into the $E$ problem and linearizing, we obtain the corresponding linearized problem:

$$A(\delta E_t) + \beta^2 \delta E_t = \omega^2 \delta \epsilon E - \nabla (\delta \epsilon \text{div} \, E) + \nabla \text{div}(\delta \epsilon E) - \delta \beta^2 E.$$
Consider now the homogeneous and perturbed $E$ problems for a specific eigenpair \((-\beta_i^2, E_i)\).

Representing the perturbation in eigenbasis $E_j$, we have:

$$\delta E_i = \sum_j (\delta E_i, E_j) E_j$$

$$A(\delta E_i) = \sum_j (\delta E_i, E_j)(-\beta_j^2) E_j$$

$$\langle A(\delta E_i), E_k \rangle = \sum_j (-\beta_j^2)(\delta E_i, E_j) \underbrace{(E_j, E_k)}_{\delta_{jk}} = (-\beta_k^2)(\delta E_i, E_k).$$

Taking the $L^2$-product of the linearized perturbed problem with $E_k$, we obtain:

$$(-\beta_i^2 - \beta_k^2)(\delta E_i, E_k) + \delta \beta_i^2 \delta_{ik} = \omega^2(\delta \epsilon E_i, E_k) - (\nabla (\delta \epsilon \text{ div } E_i), E_k) + (\nabla \text{ div } (\delta \epsilon E_i), E_k).$$

Under the assumption of distinct (simple) eigenvalues, for $k = i$, we get a formula for perturbation $\delta \beta_i^2$,

$$\delta \beta_i^2 = \omega^2(\delta \epsilon E_i, E_i) + (\delta \epsilon \text{ div } E_i, \text{ div } E_i) - (\text{div}(\delta \epsilon E_i), \text{ div } E_i).$$

For $k \neq i$, the formula allows to compute perturbation $\delta E_i$; the $i$-th component of $\delta E_i$ comes from the normalization $\|E_i + \delta E_i\| = 1$.

$$(-\beta_i^2 - \beta_k^2)(\delta E_i, E_k) = \omega^2(\delta \epsilon E_i, E_k) + (\delta \epsilon \text{ div } E_i, \text{ div } E_k) - (\text{div}(\delta \epsilon E_i), \text{ div } E_k).$$
**Linearized Mass Matrices**

Mass term \((\delta E, E)\) for different families of eigenvectors:

<table>
<thead>
<tr>
<th>((\delta E, E))</th>
<th>(E_k = \nabla \times \psi_k)</th>
<th>(E_l = \nabla \phi_l)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\delta E_i = \delta(\nabla \times \psi_i))</td>
<td>(\frac{\omega^2(\delta \epsilon E_i, E_k)}{\mu_k - \mu_i})</td>
<td>(\frac{(\omega^2 - \lambda_l)(\delta \epsilon E_i, E_l)}{\lambda_l - \mu_i})</td>
</tr>
<tr>
<td>(\delta E_j = \delta(\nabla \phi_j))</td>
<td>(\frac{\omega^2(\delta \epsilon E_j, E_k)}{\mu_k - \lambda_j})</td>
<td>(\frac{(\omega^2 - \lambda_l)(\delta \epsilon E_j, E_l) + \lambda_j \lambda_l (\delta \epsilon \phi_j, \phi_l)}{\lambda_l - \lambda_j})</td>
</tr>
</tbody>
</table>

Linearized mass matrix \((\delta E_i, E_k) + (E_i, \delta E_k)\) for different families of eigenvectors.

<table>
<thead>
<tr>
<th>((\delta E, E) + (E, \delta E))</th>
<th>(\delta E_k = \delta(\nabla \times \psi_k))</th>
<th>(\delta E_l = \delta(\nabla \phi_l))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\delta E_i = \delta(\nabla \times \psi_i))</td>
<td>0</td>
<td>not needed</td>
</tr>
<tr>
<td>(\delta E_j = \delta(\nabla \phi_j))</td>
<td>not needed</td>
<td>(-(\delta \epsilon E_j, E_l))</td>
</tr>
</tbody>
</table>
Linearized Curl-Curl Mass Matrix

We have:

\[ \delta E_i = \sum_k (\delta E_i, E_k) E_k \]  \hspace{1cm} (summation over both curls and grads)

\[ \text{curl} \delta E_i = \sum_k (\delta E_i, \nabla \times \psi_k) \mu_k \psi_k \]  \hspace{1cm} (summation over curls only.)

Hence,

\[ (\text{curl} \delta E_i, \text{curl} E_j) = (\sum_k (\delta E_i, \nabla \times \psi_k) \mu_k \psi_k, \text{curl} E_j) \]
\[ = \sum_k (\delta E_i, \nabla \times \psi_k) (\mu_k \psi_k, \mu_j \psi_j) \]
\[ = (\delta E_i, \nabla \times \psi_j) \mu_j \]

is non-zero only if \( E_j \) is a curl, \( E_j = \nabla \times \psi_j \).

Consequently, the linearized curl-curl mass matrix is equal to:

\[ (\delta E_i, E_j) \mu_j + (E_i, \delta E_j) \mu_i = \mu_j \frac{\omega^2 (\delta \epsilon E_i, E_j)}{\mu_j - \mu_i} + \mu_i \frac{\omega^2 (\delta \epsilon E_i, E_j)}{\mu_i - \mu_j} = \omega^2 (\delta \epsilon E_i, E_j) \]

if \( E_i = \nabla \times \psi_i, E_j = \nabla \times \psi_j \).
Outline

1. Eigensystems
2. Homogeneous Waveguide
3. Cylindrical Waveguide
4. Perturbation of Self-Adjoint Operators
5. Decoupling the Equations
6. Estimation
Reduction to the Second Order System

We return to the original first order system. Testing the first equation with $F_t$, and the third equation with $G_t$, we obtain:

\[
\begin{aligned}
- (i\omega E_3, \text{div } F_t) + \omega^2 (e_z \times H_t, F_t) - \frac{\partial}{\partial z} i\omega (E_t, F_t) &= i\omega (e_z \times f_t, F_t) \\
curl E_t - i\omega H_3 &= f_3 \\
-(i\omega H_3, \text{div } G_t) - \omega^2 (e_z \times \epsilon E_t, G_t) - \frac{\partial}{\partial z} i\omega (H_t, G_t) &= i\omega (e_z \times g_t, G_t) \\
curl H_t + i\omega \epsilon E_3 &= g_3.
\end{aligned}
\]

Note that, when integrating by parts the first terms, we have used the fact that $E_3 = 0$ and $n \cdot G_t = 0$ on $\partial D$. Solving the second and fourth equations in (1.1) for $E_3$ and $H_3$,

\[
\begin{aligned}
E_3 &= \frac{1}{i\omega \epsilon} g_3 - \frac{1}{i\omega \epsilon} \text{curl } H_t \\
H_3 &= -\frac{1}{i\omega} f_3 + \frac{1}{i\omega} \text{curl } E_t,
\end{aligned}
\]

and substituting into the first and the third equations, we obtain a system of two variational equations for $E_t, H_t$:

\[
\begin{aligned}
\left( \frac{1}{\epsilon} \text{curl } H_t, \text{div } F_t \right) + \omega^2 (e_z \times H_t, F_t) - \frac{\partial}{\partial z} i\omega (E_t, F_t) &= i\omega (e_z \times f_t, F_t) + \left( \frac{1}{\epsilon} g_3, \text{div } F_t \right) \\
-(\text{curl } E_t, \text{div } G_t) - \omega^2 (e_z \times \epsilon E_t, G_t) - \frac{\partial}{\partial z} i\omega (H_t, G_t) &= i\omega (e_z \times g_t, G_t) - (f_3, \text{div } G_t).
\end{aligned}
\]

(5.18)
Decoupling

Variational eigenvalue problem:

\[
\begin{aligned}
E_t & \in H_0(\text{curl}, D), H_t \in H(\text{curl}, D) \\
\left( \frac{1}{\varepsilon} \text{curl } H_t, \text{div } F_t \right) + \omega^2 (e_z \times H_t, F_t) &= -\omega \beta (E_t, F_t) \\
-\left( \text{curl } E_t, \text{div } G_t \right) - \omega^2 (e_z \times \varepsilon E_t, G_t) &= -\omega \beta (H_t, G_t) \\
F_t & \in H(\text{div}, D), \quad G_t \in H_0(\text{div}, D),
\end{aligned}
\]

is equivalent to the \( EH \) eigenproblem. Similarly, switching the role of \((E_t, H_t)\) and \((F_t, G_t)\) above, we obtain the adjoint variational eigenvalue problem equivalent to the \( FG \) eigenproblem.

We expand the unknowns into series of the perturbed eigenvectors:

\[
\begin{aligned}
E_t &= \sum_i \alpha_i E_{t1,i} + \sum_j \beta_j E_{t2,j} \\
H_t &= \sum_i \delta_i H_{t1,i} + \sum_j \eta_j H_{t2,j}
\end{aligned}
\]

where \( \alpha_i, \beta_j, \delta_i, \eta_j \) are functions of \( z \), and

\[
\begin{aligned}
E_{t1,i} &= \nabla \times \psi_i + \delta E_{t1,i}, & E_{t2,j} &= \nabla \phi_j + \delta E_{t2,j} \\
H_{t1,i} &= \nabla \psi_i + \delta H_{t1,i}, & H_{t2,j} &= \nabla \times \phi_j + \delta H_{t2,j}
\end{aligned}
\]

are the two \( E \) and \( H \) families of (perturbed) eigenvectors.
Decoupling

Let

\[ F_{t1,i} = \nabla \times \psi_i + \delta F_{t1,i}, \quad F_{t2,j} = \nabla \phi_j + \delta F_{t2,j} \]
\[ G_{t1,i} = \nabla \psi_i + \delta G_{t1,i}, \quad G_{t2,j} = \nabla \times \phi_j + \delta G_{t2,j} \]

be the corresponding families of perturbed adjoint eigenvectors.

**Scalings:**

\[
\| \nabla \times \psi_i \| = \| \nabla \psi_i \| = 1, \quad (\delta E_{t1,i}, \nabla \times \psi_i) = 0, \quad (\delta F_{t1,i}, \nabla \times \psi_i) = 0 \quad \Rightarrow \quad \| \nabla \times \psi_i + \delta E_{t1,i} \| = 1 \text{ and } (\nabla \times \psi_i + \delta E_{t1,i}, \nabla \times \psi_i + \delta F_{t1,i}) = 1.
\]

Same for the \( H \) and \( G \) eigenvectors, and the second family of eigenvectors.

Let \(-\beta^2\) be an eigenvalue for \( E \) and \( H \) eigenproblems with the corresponding eigenvectors \( E_t, H_t \) scaled as above. In order to invoke Lemma 1 (b) argument, we have to replace \( H_t \) with \( cH_t \) where constant \( c \) is computed by comparing eigenvector \( cH_t \) with \( H_t \) given by relation (1.6),

\[ cH_t = \frac{1}{\omega \beta} \left[ -\nabla \text{curl} E_t + \omega^2 e_z \times \epsilon E_t \right]. \]

Pair \((E_t, cH_t)\) constitutes then an eigenvector for system (1.4) corresponding to root \( \beta \) of \( \beta^2 \)

selected in such a way that \( e^{i\beta z} \) represents an outgoing wave. We proceed similarly with the

adjoint eigenvectors. Let \(-\gamma^2\) be an eigenvalue for problems (1.11) and (1.13) with the

corresponding eigenvectors \( F_t, H_t \). After scaling the second component, pair \((F_t, dG_t)\) constitutes an eigenvector for system (1.5) corresponding to a root \( \gamma \) of \( \gamma^2 \).
Decoupling

Case: $\beta^2 \neq \gamma^2$ and, therefore, $\beta \neq \gamma$. Testing the 2nd order system with pair $(F_t, G_t)$, we obtain the bi-orthogonality condition,

$$c(BH_t, F_t) + d(CE_t, G_t) = 0$$

where $B$ and $C$ denote the operators on the left-hand side of the system. But, testing with the adjoint eigenpair $(F_t, -G_t)$ (corresponding to eigenvalue $-\gamma \neq \beta$), we obtain also

$$c(BH_t, F_t) - d(CE_t, G_t) = 0.$$ 

Consequently,

$$(BH_t, F_t) = 0 \quad \text{and} \quad (CE_t, G_t) = 0.$$ 

Case: $\beta^2 = \gamma^2$ and $\beta = \gamma$. Testing with pair $(F_t, dH_t)$, we obtain:

$$c(BH_t, F_t) + d(CE_t, G_t) = \omega \beta [1 + cd].$$ 

But, testing with the adjoint eigenpair $(F_t, -H_t)$ (corresponding to eigenvalue $-\gamma \neq \beta$), we obtain also

$$c(BH_t, F_t) - d(CE_t, G_t) = 0.$$ 

Consequently,

$$(BH_t, F_t) = -\frac{\omega \beta}{2c} [1 + cd] =: \theta \quad \text{and} \quad (CE_t, G_t) = -\frac{\omega \beta}{2d} [1 + cd] =: \nu.$$
Decoupling

Theorem

Testing with \((F_{t1,j}, G_{t1,j})\) and with \((F_{t2,j}, G_{t2,j})\) we obtain a decoupled system of ODEs for the coefficients \(\alpha_j, \delta_j\):

\[
\begin{align*}
\theta_{1,j} \delta_j - i \omega \alpha_j' &= r_1(z) := (i \omega e_z \times f_t, F_{t1,j}) + \left( \frac{1}{\epsilon} g_3, \text{div} F_{t1,j} \right) \\
\nu_{1,j} \alpha_j - i \omega \delta_j' &= r_2(z) := (i \omega e_z \times g_t, G_{t1,j}) - (f_3, \text{div} G_{t1,j})
\end{align*}
\]

and \(\beta_j, \eta_j\):

\[
\begin{align*}
\theta_{2,j} \eta_j - i \omega \beta_j' &= s_1(z) := (i \omega e_z \times f_t, F_{t2,j}) + \left( \frac{1}{\epsilon} g_3, \text{div} F_{t2,j} \right) \\
\nu_{2,j} \beta_j - i \omega \eta_j' &= s_2(z) := (i \omega e_z \times g_t, G_{t2,j}) - (f_3, \text{div} G_{t2,j})
\end{align*}
\]

where

\[
\begin{align*}
\theta_{1,j} &= -\omega^2, & \nu_{1,j} &= -\beta_j^2 - \omega^2 (\delta \epsilon \nabla \psi_j, \nabla \psi_j) \\
\theta_{2,j} &= \beta_j^2 + \lambda_j^2 (\delta \epsilon \phi_j, \phi_j) & \nu_{2,j} &= \omega^2 + \omega^2 (\delta \epsilon \nabla \phi_j, \nabla \phi_j).
\end{align*}
\]

Watch out for the terrible notational collision with \(\beta\)'s.
1. Eigensystems
2. Homogeneous Waveguide
3. Cylindrical Waveguide
4. Perturbation of Self-Adjoint Operators
5. Decoupling the Equations
6. Estimation
Estimation of $E_t$

\[ \|E_t\|^2 \leq 2 \left[ \| \sum_{i=1}^{\infty} \alpha_i E_{t1,i} \|^2 + \| \sum_{j=1}^{\infty} \beta_j E_{t2,j} \|^2 \right] \]

\[ = 2 \lim_{N \to \infty} \left[ \sum_{i=1}^{N} \alpha_i E_{t1,i} + \sum_{k=1}^{N} \alpha_k E_{t1,k} \right] + \left[ \sum_{j=1}^{N} \beta_j E_{t2,j} + \sum_{i=1}^{N} \beta_i E_{t2,i} \right] \]

\[ = 2 \lim_{N \to \infty} \left[ \sum_{i,k=1}^{N} \alpha_i \overline{\alpha_k} (E_{t1,i}, E_{t1,k}) + \sum_{j,l=1}^{N} \beta_j \overline{\beta_l} (E_{t2,j}, E_{t2,l}) \right] \]

\[ \leq \lim_{N \to \infty} C \left[ \sum_{i=1}^{N} |\alpha_i|^2 + \sum_{j=1}^{N} |\beta_j|^2 \right] \]

\[ = C \left[ \sum_{i=1}^{\infty} |\alpha_i|^2 + \sum_{j=1}^{\infty} |\beta_j|^2 \right] \]

where (up to linearization) $C = 2(1 + \|\delta\epsilon\|_{L^\infty(D)})$. After integrating in $z$, we get

\[ \int_0^L \|E_t\|^2 \, dz \leq C \left[ \sum_{i=1}^{\infty} \int_0^L |\alpha_i|^2 \, dz + \int_0^L |\beta_j|^2 \, dz \right]. \]
Estimation of $\text{curl } E_t$

$$\| \text{curl } E_t \|^2 \leq 2 \left[ \| \sum_{i=1}^{\infty} \alpha_i \text{curl } E_{t1,i} \|^2 + \| \sum_{j=1}^{\infty} \beta_j \text{curl } E_{t2,i} \|^2 \right]$$

$$= 2 \lim_{N \to \infty} \left[ (\sum_{i=1}^{N} \alpha_i \text{curl } E_{t1,i}, \sum_{k=1}^{N} \alpha_k \text{curl } E_{t1,k}) + (\sum_{j=1}^{N} \beta_j \text{curl } E_{t2,i}, \sum_{l=1}^{N} \beta_l \text{curl } E_{t2,i}) \right]$$

$$= 2 \lim_{N \to \infty} \left[ \sum_{i,k=1}^{N} \alpha_i \overline{\alpha_k} (\text{curl } E_{t1,i}, \text{curl } E_{t1,k}) + \sum_{j,l=1}^{N} \beta_j \overline{\beta_l} (\text{curl } E_{t2,j}, \text{curl } E_{t2,l}) \right]$$

$$\approx 2 \sum_{i=1}^{\infty} (\mu_i + \omega^2 \| \delta \epsilon \|_{L^\infty(D)}) |\alpha_i|^2 .$$

Note that, like for the homogeneous case, the perturbed gradients do not contribute (the linearized perturbed curl mass matrix is zero).
Estimation of Coefficients $\alpha_i, \delta_i$

We focus now on the ODE boundary-value problem for coefficients $\alpha$ and $\delta$,

$$
\begin{align*}
\alpha(0) &= 0, \quad \alpha(L) = \frac{\nu}{\theta} \delta(L) \\
\theta \delta - i \omega \alpha' &= r_1 \\
\nu \alpha - i \omega \delta' &= r_2 .
\end{align*}
$$

Testing the second equation with $\delta \alpha$, $\delta \alpha(0) = 0$, integrating the derivative term by parts, and utilizing BC, we obtain:

$$
i \omega (\delta, \delta \alpha') = -\omega \beta(\alpha, \delta \alpha) + i \omega \alpha(L) \delta \alpha(L) + (r_2, \delta \alpha) .
$$

Testing now the first equation with $\delta \alpha'$ and using the formula above, we obtain the ultimate variational problem for coefficient $\alpha$,

$$
\begin{align*}
\alpha(0) &= 0 \\
(\alpha', \delta \alpha') - \kappa^2 (\alpha, \delta \alpha) + \kappa \alpha(L) \delta \alpha(L) &= \frac{1}{\omega} (r_1, \delta \alpha') - \frac{\beta}{\omega} (r_2, \delta \alpha) \\
\forall \delta \alpha : \delta \alpha(0) &= 0
\end{align*}
$$

where $\kappa = i \frac{\sqrt{\theta \nu}}{\omega}$.
Lemma (7)

Let $I = (0, L)$. Consider two problems: Find $q_1, q_2 \in H^1_0(I) := \{ w \in H^1(I) : w(0) = 0 \}$ such that

\[
(q'_1, w') + \beta^2 (q_1, w) + \beta q_1(L)w(L) = (f, w) \quad w \in H^1_0(I),
\]

\[
(q'_2, w') + \beta^2 (q_2, w) + \beta q_2(L)w(L) = (f, w') \quad w \in H^1_0(I).
\]

(i) If $\beta \in i\mathbb{R}$ then,

\[
\|q'_1\|^2 + \beta^2 \|q_1\|^2 \leq CL^2\|f\|^2,
\]

\[
\|q'_2\|^2 + \beta^2 \|q_2\|^2 \leq CL^2|\beta|^2\|f\|^2,
\]

where $C > 0$ depend only on a lower bound for $L|\beta|$. 

(ii) If $\beta > 0$ then,

\[
\|q'_1\|^2 + \beta^2 \|q_1\|^2 \leq C\beta^{-2}\|f\|^2,
\]

\[
\|q'_2\|^2 + \beta^2 \|q_2\|^2 \leq C\|f\|^2,
\]

where $C > 0$ depend only on a lower bound for $L\beta$. 

1D Stability Result
Estimation of $\alpha_j$

Term 1: $i\omega(e_z \times f_t, F_{t1,j})$ contributing to $r_1$. Skipping factor $i\omega$, we have:

$$\sum_j \int_0^L |\alpha_j|^2 \, dz \lesssim \sum_j \int_0^L \beta_j^{-2} |(e_z \times f_t, F_{t1,j} + \delta F_{t1,j})|^2$$

(Lemma 7 (ii)\_1)

$$\lesssim 2 \sum_j \int_0^L \left[ |(e_z \times f_t, F_{t1,j})|^2 + |(e_z \times f_t, \delta F_{t1,j})|^2 \right]$$

(Young’s inequality)

$$\lesssim 2 \sum_j \int_0^L |(e_z \times f_t, F_{t1,j})|^2$$

(linearization)

$$\leq 2 \int_0^L \|e_z \times f_t\|^2 \, dz$$

$$= 2 \int_0^L \|f_t\|^2 \, dz.$$

Term 2: $(\frac{1}{\varepsilon} g_3, \text{div } G_t)$ contributing to $r_1$.

$$\sum_j \int_0^L |\alpha_j|^2 \, dz \lesssim \sum_j \int_0^L \beta_j^{-2} |(\frac{1}{\varepsilon} g_3, \text{div}(F_{t1,j} + \delta F_{t1,j}))|^2$$

(Lemma 7 (ii)\_1)

$$\leq 2 \sum_j \int_0^L \beta_j^{-2} \left[ |(\frac{1}{\varepsilon} g_3, \text{div}(F_{t1,j}))|^2 + |(\frac{1}{\varepsilon} g_3, \text{div}(\delta F_{t1,j}))|^2 \right]$$

(Young’s lemma)

$$\lesssim 2 \sum_j \int_0^L \beta_j^{-4} |(\frac{1}{\varepsilon} g_3, \text{div}(F_{t1,j}))|^2$$

(linearization)

$$\lesssim 0$$

(div $F_{t1,j} = 0$)
Estimation of $\alpha_j$ - Cont.

Term 3: $i\omega(e_z \times f_t, G_{t1,j})$ contributing to $r_2$. We follow exactly the same reasoning as for Term 1, sparing a factor $\beta_j^{-2}$.

Term 4: $(f_3, \text{div } G_{t1,j})$ contributing to $r_2$.

$$
\sum_j \int_0^L |\alpha_j|^2 \, dz \lesssim \sum_j \int_0^L \beta_j^{-2} |(f_3, \text{div}(G_{t1,j} + \delta G_{t1,j}))|^2 \\
\lesssim 2 \sum_j \int_0^L \beta_j^{-2} [|(f_3, \text{div}(G_{t1,j}))|^2 + |(f_3, \text{div}(\delta G_{t1,j}))|^2] \\
\lesssim 2 \sum_j \int_0^L \beta_j^{-2} |(f_3, \text{div}(G_{t1,j}))|^2 \\
\lesssim 2 \sum_j \int_0^L \beta_j^{-2} \mu_j |(f_3, \mu_j^{1/2} \psi_j)|^2 \\
\lesssim 2 \sum_j |(f_3, \mu_j^{1/2} \psi_j)|^2 = 2 \|f_3\|_2.
$$

(Lemma 7 (ii))

(Young’s lemma)

(linearization)

(\(\beta_j^{-2} \mu_j \approx O(1)\))

Estimation of curl $E_t$.

We need to estimate:

$$
\sum_i \int_0^L (\mu_i + \|\delta\epsilon\|_{L^\infty(D)}) |\alpha_i|^2 \, dz. \\
\sim \beta_i^2
$$

We follow exactly the same strategy as above. In all cases, we can accommodate the extra $\beta_i^2$ factor.
Final Result

We follow the same reasoning for the remaining coefficients $\delta_j, \beta_j, \eta_j$ to arrive at our final result.

**Theorem**

Let $\Omega = D \times (0, L)$. Assume that the dielectric constant $\epsilon$ is a sufficiently\(^a\) small perturbation of a constant. There exists then a constant $C > 0$, independent of data $f$, $g$ and solution $E$, $H$ such that

$$\|E\|_{L^2(\Omega)}^2 + \|H\|_{L^2(\Omega)}^2 \leq CL^2 \left( \|f\|_{L^2(\Omega)}^2 + \|g\|_{L^2(\Omega)}^2 \right).$$

\(^a\)So that the perturbation technique based on linearization is justified.

Thank you for your attention!
References
