Matrix-valued finite elements

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In collaboration with
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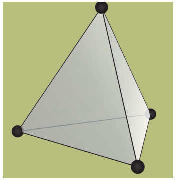
Thanks: DMS

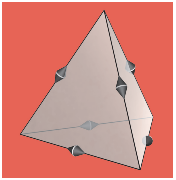
Outline

Lowest order scalar ($\mathbb{R}$) and vector ($\mathbb{V}$) valued finite elements

($\mathbb{R}$-valued)



($\mathbb{V}$-valued)



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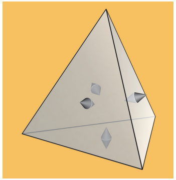


($\mathbb{R}$-valued)

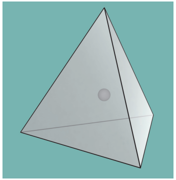
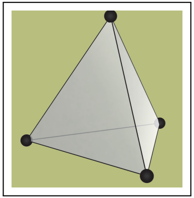


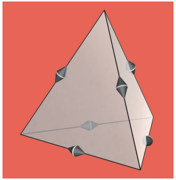
Figure Source: [Arnold+Logg 2014] poster "The Periodic Table of Finite Elements."

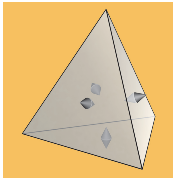
Lowest order scalar ($\mathbb{R}$) and vector ($\mathbb{V}$) valued finite elements

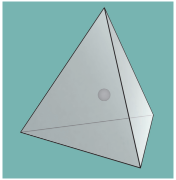
($\mathbb{R}$-valued) ($\mathbb{V}$-valued) ($\mathbb{V}$-valued) ($\mathbb{R}$-valued)



grad, continuous







Lagrange finite element space [Courant 1943]

Natural operator: grad

Continuous across elements

Figure Source: [Arnold+Logg 2014] poster "The Periodic Table of Finite Elements."

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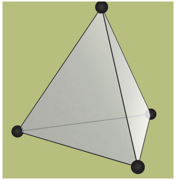
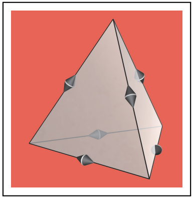
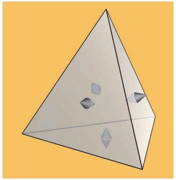
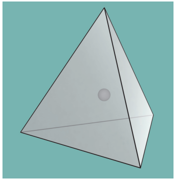
($\mathbb{R}$-valued)	($\mathbb{V}$-valued)	($\mathbb{V}$-valued)	($\mathbb{R}$-valued)
			
grad, continuous	curl, "t-continuous"		

Nédélec finite element space [Nédélec 1980]

Natural operator: curl

Inter-element continuity: t-component

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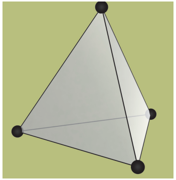
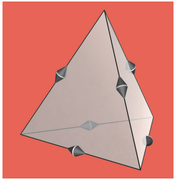


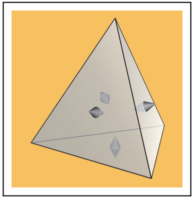
grad, continuous

($\mathbb{V}$-valued)



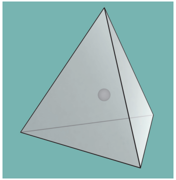
curl, "t-continuous"

($\mathbb{V}$-valued)



div, "n-continuous"

($\mathbb{R}$-valued)



Raviart-Thomas finite element space [Raviart+Thomas 1977]

Natural operator: div

Inter-element continuity: n-component

Figure Source: [Arnold+Logg 2014] poster "The Periodic Table of Finite Elements."

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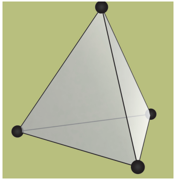
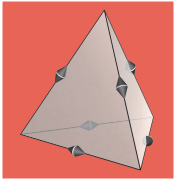
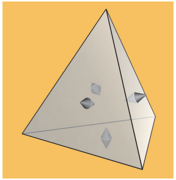


grad, continuous



curl, "t-continuous"



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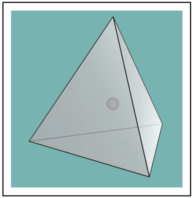

Discontinuous Galerkin finite element space

Inter-element continuity: None

Figure Source: [Arnold+Logg 2014] poster "The Periodic Table of Finite Elements."

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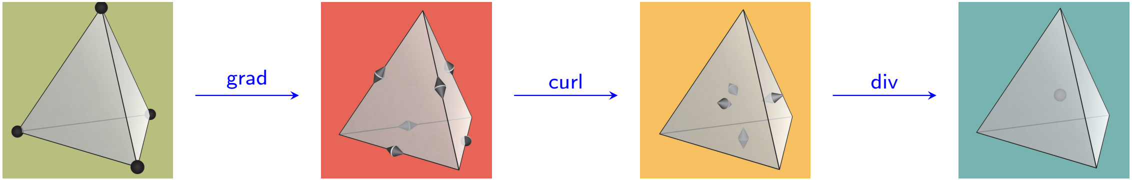


grad, continuous | curl, "t-continuous" | div, "n-continuous"

These finite element spaces are now collectively understood using the de Rham complex.

Where do matrix-valued finite elements fit? And second-order differential operators?

<u>Next example:</u> curl div *operator and "nt-continuous" matrix elements for viscous stresses.*

Stokes system

Textbook version:

$$\begin{aligned} -\nu\Delta u + \nabla p &= f && \text{in } \Omega \\ \operatorname{div} u &= 0 && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega \end{aligned}$$

Physical stress-based version:

$$\begin{aligned} \frac{1}{2\nu}\sigma - \varepsilon(u) &= 0 && \text{in } \Omega \\ \operatorname{div}\sigma - \nabla p &= -f && \text{in } \Omega \\ \operatorname{div} u &= 0 && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega \end{aligned}$$

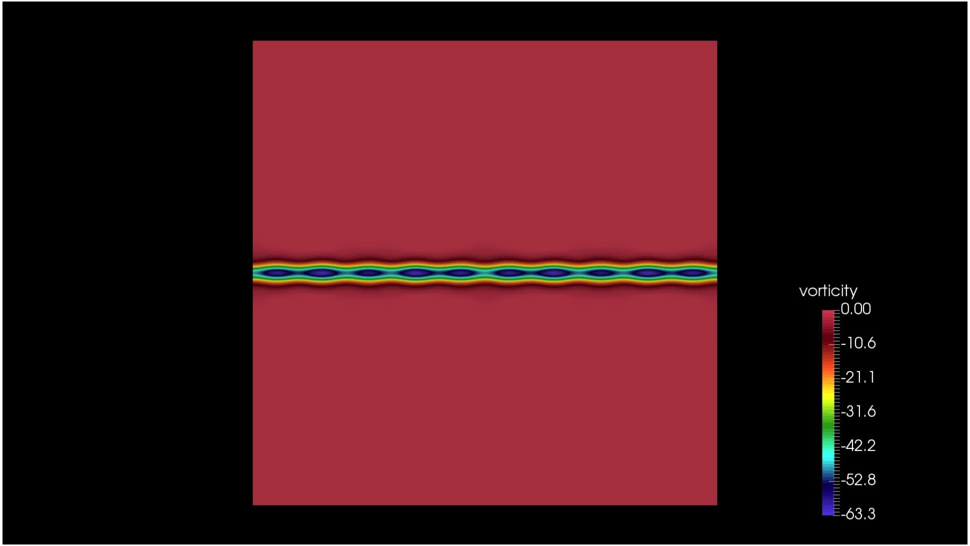
Notation:

- $\nu(x)$ = kinematic viscosity
- $f(x)$ = body force
- $u(x)$ = fluid velocity
- $p(x)$ = kinematic pressure
- $\sigma(x)$ = viscous stress
- $\varepsilon(u) = \frac{1}{2}(\nabla u + (\nabla u)') = \operatorname{sym}(\nabla u)$.

What topology on σ pairs well with $H(\mathrm{div})$-topology on u?

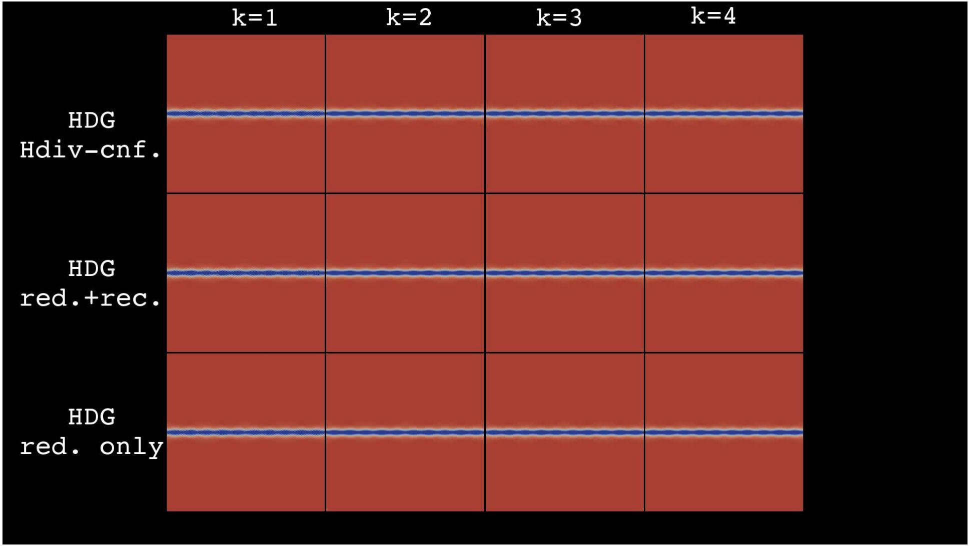
Outline

Kelvin-Helmholtz instability simulation



[Schroeder+John+Lederer+Lehrenfeld+Lube+Schöberl 2019]

Different results from different discretizations



Source: P. Lederer's thesis

Mass conservation

Mass is neither created nor destroyed.

- Mass $= \displaystyle\int \rho$, density integrated.
- Incompressibility $\implies$ the material derivative $\dfrac{D\rho}{Dt} = 0$.

Equivalent constraint on fluid velocity u:

- Continuity equation (mass conservation) $\implies$ $\dfrac{D\rho}{Dt} + \rho \operatorname{div} u = 0$.
- Hence, mass conservation in incompressible fluids $\iff$ $\boxed{\operatorname{div} u = 0}$.

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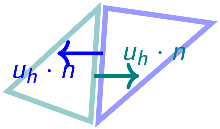
A piecewise polynomial fluid velocity approximation u_h has $\boxed{\operatorname{div} u_h = 0}$ if

- its divergence is zero pointwise within elements,
- and $[\![u_h \cdot n]\!] = 0$, i.e., u_h has at least n-continuity.



How can methods gain exact mass conservation?

Discrete version of the incompressibility constraint in a space V_h with n-continuity:

$$(\operatorname{div} u_h,\, q_h) = 0 \qquad \text{for all } q_h \text{ in the discrete pressure space } Q_h.$$

Hence u_h is exactly mass conserving if $\boxed{\operatorname{div} V_h \subseteq Q_h.}$

Two approaches to methods with mass conservation:

$\underline{H(\text{div})\text{-based}}$ $\underline{H^1\text{-based}}$

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Two approaches to methods with mass conservation:

<u>$H(\operatorname{div})$-based</u>

$$H(\operatorname{div}) = \{u :\ u_i \in L_2,\ \textstyle\sum_i \partial_i u_i \in L_2\}$$

<u>H^1-based</u>

$$H^1 \otimes \mathbb{V} = \{u :\ u_i \in L_2,\ \partial_j u_i \in L_2\}$$

(This is *vector* H^1.)

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Two approaches to methods with mass conservation:

$H(\text{div})$-based

When $V_h \subset H(\text{div})$:

- $\oplus$ Range of divergence is simple.
- $\ominus$ Discretizing viscous term is complex.

H^1-based

When $V_h \subset H^1 \otimes \mathbb{V}$:

- $\ominus$ Range of divergence is complex.
- $\oplus$ Viscous term easily discretized.

How can methods gain exact mass conservation?

Discrete version of the incompressibility constraint in a space V_h with n-continuity:

$$(\operatorname{div} u_h, q_h) = 0 \qquad \text{for all } q_h \text{ in the discrete pressure space } Q_h.$$

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Two approaches to methods with mass conservation:

$\underline{H(\text{div})\text{-based}}$

[Cockburn+Kanschat+Schötzau 2005]

$\vdots$

[Cockburn+G+Nguyen+Peraire+Sayas 2011]

[Linke+Merdon 2016]

[G+Lederer+Schöberl 2019] (MCS method)

$\underline{H^1\text{-based}}$

[Scott+Vogelius 1985]

$\vdots$

[Guzmán+Scott 2019]

[Neilan 2020]

[Ainsworth+Parker 2021]

Punchlines of Mass-Conserving Stress-yielding (MCS) method

- *Structure-preservation:*
  Numerical solutions are exactly mass conserving.
  Scheme is pressure robust.

- *Viscous stress σ approximation:* New matrix-valued finite elements with *continuous shear component* (or nt-component) used for approximating σ. $(\llbracket t \cdot \sigma\, n \rrbracket \equiv \llbracket \sigma_{nt} \rrbracket = 0.)$

- *Optimally convergent in* velocity (u), pressure (p), viscous stress (σ), and vorticity (ω).

- *More features:*
  - No stabilization parameters.
  - Easy to incorporate stress boundary conditions.
  - No vertex unknowns, facet couplings only: permits easy hybridization.
  - Element mappings (even curvilinear) are straightforward.
  - Stable for ∇u-forms as well as the more physical $\varepsilon(u)$-formulations.

Deriving the MCS formulation

Standard form:	With σ:	
$-\nu\Delta u + \operatorname{grad} p = 0$	$\frac{1}{2\nu}\sigma - \varepsilon(u) = 0$	in Ω
	$\operatorname{div}\sigma - \operatorname{grad} p = -f$	in Ω
$\operatorname{div} u = 0$	$\operatorname{div} u = 0$	in Ω
$u = 0$	$u = 0$	on $\partial\Omega$

Deriving the MCS formulation

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\operatorname{div} u &= 0 \\
u &= 0
\end{aligned}
$$

With σ:

$$
\begin{aligned}
\tfrac{1}{2\nu} \operatorname{dev} \sigma - \varepsilon(u) &= 0 && \text{in } \Omega \\
\operatorname{div} \sigma - \operatorname{grad} p &= -f && \text{in } \Omega \\
\operatorname{div} u &= 0 && \text{in } \Omega \\
u &= 0 && \text{on } \partial\Omega
\end{aligned}
$$

Notation:

$$
\operatorname{dev} m := m - \frac{1}{3}(\mathrm{tr}\, m)\delta, \qquad m \text{ in } \mathbb{R}^{3\times 3}
$$

$$
\sigma = \operatorname{dev} \sigma, \text{ in } \mathbb{T} \qquad \mathbb{T} := \operatorname{dev}(\mathbb{R}^{3\times 3}) \equiv \{\text{traceless matrices}\}
$$

Deriving the MCS formulation

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$\operatorname{div} u = 0$	$\operatorname{div} u = 0$	in Ω
	$\sigma - \sigma' = 0$	in Ω
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$$\sigma = \operatorname{dev}\sigma, \text{ in } \mathbb{T} \qquad \mathbb{T} := \operatorname{dev}(\mathbb{R}^{3\times 3}) \equiv \{\text{traceless matrices}\}$$

$$\omega := \text{ vorticity, in } \mathbb{K} \qquad \mathbb{K} := \operatorname{skw}(\mathbb{R}^{3\times 3}) \equiv \{\text{skew-symmetric matrices}\}$$

Deriving the MCS formulation

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With σ:

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\end{aligned}
$$

Test with $\tau : \Omega \to \mathbb{T}$, $v : \Omega \to \mathbb{V}$, $q : \Omega \to \mathbb{R}$, $\eta : \Omega \to \mathbb{K}$:

$$
\boxed{
\begin{aligned}
(\tfrac{1}{2\nu}\operatorname{dev}\sigma, \operatorname{dev}\tau) + (u, \operatorname{div}\tau) + (\omega, \tau) &= 0 \\
(\operatorname{div}\sigma, v) + (p, \operatorname{div} v) &= -(f, v) \\
(\operatorname{div} u, q) &= 0 \\
(\sigma, \eta) &= 0.
\end{aligned}
}
$$

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Test with $\tau : \Omega \to \mathbb{T}$, $v : \Omega \to \mathbb{V}$, $q : \Omega \to \mathbb{R}$, $\eta : \Omega \to \mathbb{K}$:

$$\boxed{\begin{aligned} \left(\tfrac{1}{2\nu}\operatorname{dev}\sigma, \operatorname{dev}\tau\right) + (u, \operatorname{div}\tau) + (\omega, \tau) &= 0 \\ (\operatorname{div}\sigma, v) + (p, \operatorname{div} v) &= -(f, v) \\ (\operatorname{div} u, q) &= 0 \\ (\sigma, \eta) &= 0. \end{aligned}}$$

Derivation is completed by replacing the integrals $(\operatorname{div}\sigma, v)$ by functional actions $\langle \operatorname{div}\sigma, v\rangle$.

The viscous stress space

MCS formulation: Find $\sigma \in \Sigma$, $u \in \mathring{H}(\mathrm{div})$, $p \in L_{2,\mathbb{R}}$, $\omega \in L_2 \otimes \mathbb{K}$ satisfying

$$\boxed{\begin{aligned} (\tfrac{1}{2\nu}\operatorname{dev}\sigma, \operatorname{dev}\tau) + \langle \operatorname{div}\tau, u\rangle + (\omega, \tau) &= 0, && \tau \in \Sigma, \\ \langle \operatorname{div}\sigma, v\rangle + (p, \operatorname{div} v) &= -\langle f, v\rangle, && v \in \mathring{H}(\mathrm{div}), \\ (\operatorname{div} u, q) &= 0, && q \in L_{2,\mathbb{R}}, \\ (\sigma, \eta) &= 0, && \eta \in L_2 \otimes \mathbb{K}. \end{aligned}}$$

Minimal requirement for any σ in Σ is $\operatorname{div}\sigma \in \mathring{H}(\mathrm{div})^*$, the dual space of $\mathring{H}(\mathrm{div})$.

Theorem: $\mathring{H}(\mathrm{div})^* = H^{-1}(\mathrm{curl})$.

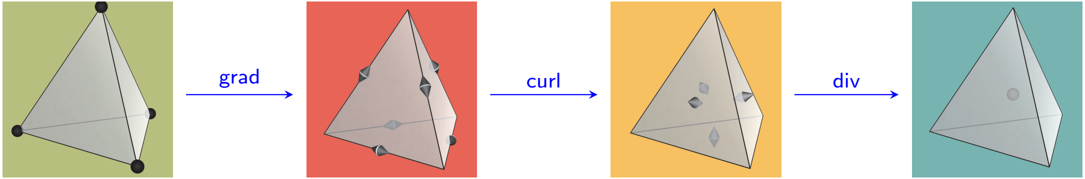
$$\boxed{\textit{Space for viscous fluid stresses: } \Sigma = \left\{\tau \in L_2 \otimes \mathbb{T} : \ \operatorname{curl}\operatorname{div}(\tau) \in H^{-1}\right\}.}$$

Notation: ❶ $\mathbb{T} = \{\text{traceless matrices}\}$, ❷ $\mathbb{K} = \{\text{skew symmetric matrices}\}$, ❸ $L_{2,\mathbb{R}} = \{p \in L_2 : (p, 1) = 0\}$, ❹ $H^{-1}(\mathrm{curl}) = \{\phi \in H^{-1} \otimes \mathbb{V} : \ \operatorname{curl}\phi \in H^{-1} \otimes \mathbb{V}\}$, ❺ $\mathring{H}(\mathrm{div}) = \{w \in H(\mathrm{div}) : w \cdot n|_{\partial\Omega} = 0\}$.

Outline

The *nt*-continuity

($\mathbb{R}$-valued) ($\mathbb{V}$-valued) ($\mathbb{V}$-valued) ($\mathbb{R}$-valued)



grad, continuous $[\![w]\!] = 0$

curl, "t-continuous" $[\![t \cdot u]\!] = 0$

div, "n-continuous" $[\![n \cdot q]\!] = 0$

For the curl div *operator, we make "nt-continuous" matrix elements for viscous stresses,* i.e.,

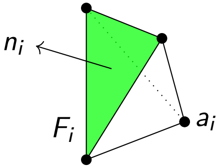
$$[\![t \cdot \sigma n]\!] = 0$$

for any tangent vector t and normal vector n on element boundaries.

Designing finite elements for viscous stresses

Lowest order case: *Is it possible to construct a constant matrix function whose nt-component vanishes on ∂T except on one face?*

<u>Tetrahedron T</u>



- Consider $\sigma \;=\; n_j \otimes (n_k \times n_l)$

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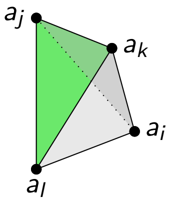

- Consider $\sigma \quad = \qquad n_j \otimes (n_k \times n_l)$
- $\sigma n = 0$ on $F_k \cup F_l$

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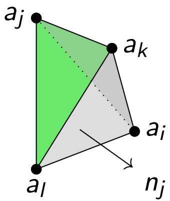

- Consider $\sigma \quad = \qquad n_j \otimes (n_k \times n_l)$
- $\sigma n = 0$ on $F_k \cup F_l$ and σn_j is collinear with n_j, so $\sigma n_j \cdot t|_{F_j} = 0$.

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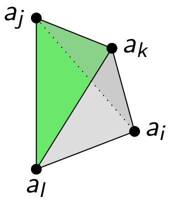
Tetrahedron T

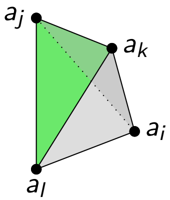


- Consider $\sigma \quad = \qquad n_j \otimes (n_k \times n_l)$
- Thus $t \cdot \sigma n = 0$ on all faces except F_i.

Designing finite elements for viscous stresses

Lowest order case: *Is it possible to construct a constant matrix function whose nt-component vanishes on ∂T except on one face?*

Tetrahedron T



- Consider $\sigma = \text{dev}[n_j \otimes (n_k \times n_l)]$
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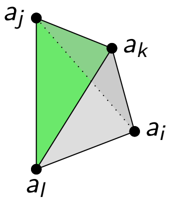

- Consider $\sigma \;\; = \text{dev}[n_j \otimes (n_k \times n_l)]$ and $\sigma = \text{dev}[n_l \otimes (n_j \times n_k)]$.
- Thus $t \cdot \sigma n = 0$ on all faces except F_i.

Designing finite elements for viscous stresses

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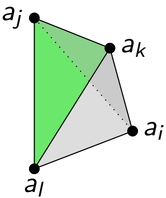
<u>Tetrahedron T</u>



- Consider $\sigma \;\; = \operatorname{dev}[n_j \otimes (n_k \times n_l)]$ and $\sigma = \operatorname{dev}[n_l \otimes (n_j \times n_k)]$.
- Taking all faces, we get a basis for $\mathbb{T} = \{m \in \mathbb{R}^{3\times 3} : \operatorname{tr} m = 0\}$.

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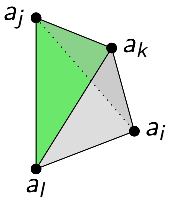

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- Taking all faces, we get a basis for $\mathbb{T} = \{m \in \mathbb{R}^{3\times3} : \operatorname{tr} m = 0\}$.

More general shape functions: $\lambda_i^{\alpha_i}\lambda_j^{\alpha_j}\lambda_k^{\alpha_k}\,\text{dev}[\nabla\lambda_i \otimes (\nabla\lambda_j \times \nabla\lambda_k)]$.

Unisolvent finite element

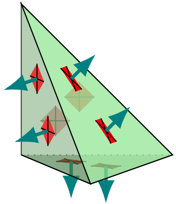
Ciarlet-style finite element definition:

- *Geometry:* Tetrahedron T
- *Space:* $P_k(T) \otimes \mathbb{T} = \{m : T \to \mathbb{T} \big| \; m_{ij}$ is a polynomial of degree $\leq k\}$ for any $k \geq 0$.
- *Degrees of freedom:*

$$\int_F \sigma_{nt} \cdot r, \qquad \text{tangential } r \text{ with } r_i \in P_k(F) \text{ on each face } F,$$

$$\int_T \sigma : \varsigma, \qquad \varsigma \in P_{k-1}(T) \otimes \mathbb{T}, \quad \text{if } k \geq 1.$$

Theorem: These degrees of freedom are unisolvent for $P_k(T) \otimes \mathbb{T}$. They generate global degrees of freedom suitable for enforcing nt-continuity. [G+Lederer+Schöberl 2019]

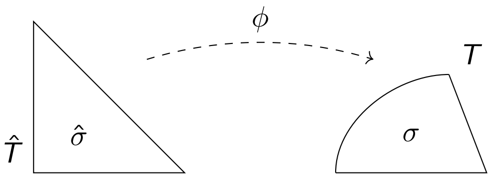


The $k = 0$ case

Mapping

- Let T = image of reference element $\hat{T}$ under (possibly curvilinear) map ϕ,
- and define the curl div pullback by

$$\hat{\sigma} = \det(\nabla\phi)\, (\nabla\phi)^t\, (\sigma \circ \phi)\, (\nabla\phi)^{-t}.$$



Theorem: Let $F = \phi(\hat{F})$ for a facet $\hat{F}$ of $\hat{T}$ with normal $\hat{n}$ and $\hat{t} \in \hat{n}^{\perp}$. Let n and t be the mapped normal and tangent on F. Then

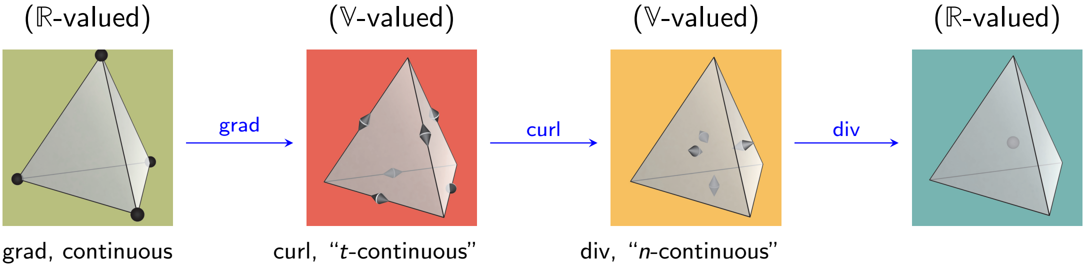
- *There is a nonzero c_F depending only on $\phi|_{\hat{F}}$ such that $\hat{t} \cdot \hat{\sigma}\, \hat{n} = c_F\, (t \cdot \sigma\, n) \circ \phi$,*
- *The mapped $\hat{\sigma}$ is traceless if and only if σ is.*

Other MCS variables have standard pullbacks (e.g., standard Piola for $H(\mathrm{div})$-velocity).

Thus all elements used in the MCS method have natural mappings.

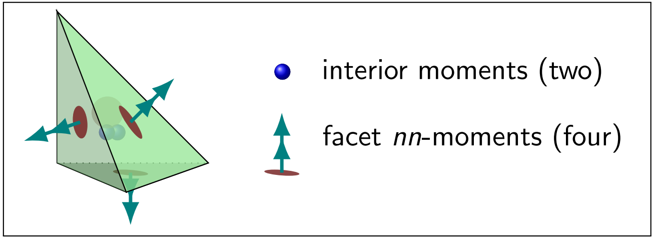
Outline

Where do matrix-valued finite elements fit?



- Where does the *nt*-continuous $\mathbb{T}$-valued matrix element fit?
- How about matrix elements with *tt*-continuity?
- How about matrix elements with *nn*-continuity?

More matrix-valued elements



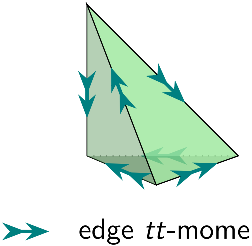
Coupling dofs:

$$\sigma \mapsto \int_F n \cdot \sigma n$$

$\mathbb{S}$-valued HHJ element, aka TDNNS element [Hellan 1967] [Herrmann 1967] [Johnson 1973]
Natural operator: div div [Comodi 1989] [Pechstein+Schöberl 2011]
Continuous *nn*-component

Notation: ❶ $\text{sym}(\tau) = \frac{1}{2}(\tau + \tau^\top)$ for matrices $\tau \in \mathbb{R}^{3\times 3}$. ❷ $\mathbb{S} = \text{sym}(\mathbb{R}^{3\times 3})$, space of symmetric matrices.

More matrix-valued elements



Coupling dofs on each edge:

$$g \mapsto \int_E t \cdot g t$$

$\mathbb{S}$-valued Regge element

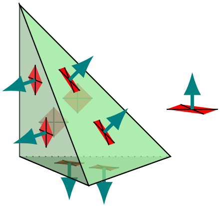
Natural operator: curl $\top$ curl or "inc"

Continuous tt-component

[Regge 1961] [Christiansen 2011] [Li 2018]
[G+Neunteufel+Schöberl+Wardetsky 2023]

Notation: ① $\mathrm{sym}(\tau) = \frac{1}{2}(\tau + \tau^\top)$ for matrices $\tau \in \mathbb{R}^{3\times 3}$. ② $\mathbb{S} = \mathrm{sym}(\mathbb{R}^{3\times 3})$, space of symmetric matrices.

More matrix-valued elements



Two coupling dofs on each face:

$$\tau \mapsto \int_F t \cdot \tau n$$

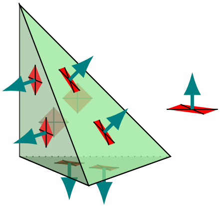
$\mathbb{T}$-valued viscous stress element [G+Lederer+Schöberl 2019]

Natural operator: curl div

Continuous nt-component

Notation: ① $\operatorname{sym}(\tau) = \frac{1}{2}(\tau + \tau^\top)$ for matrices $\tau \in \mathbb{R}^{3\times 3}$. ② $\mathbb{S} = \operatorname{sym}(\mathbb{R}^{3\times 3})$, space of symmetric matrices. ③ $\operatorname{dev}\tau = \tau - \frac{1}{3}(\operatorname{tr}\tau)I$ for matrices τ, ④ $\mathbb{T} = \operatorname{dev}\mathbb{R}^{3\times 3}$ = space of traceless matrices.

More matrix-valued elements



facet nt-moments

Two coupling dofs on each face:

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$\mathbb{T}$-valued viscous stress element [G+Lederer+Schöberl 2019]

Natural operator: curl div

Continuous nt-component

Recent understanding: An algebraic structure with matrix-valued Sobolev space functions connected by curl div, curl $\top$ curl, div div.

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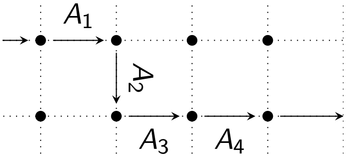
Outline

Complexes from the 2-complex

We use "commutative diagrams"

- whose "vertices" or "objects" are function spaces,
- whose "arrows" or "morphisms" are differential operators.



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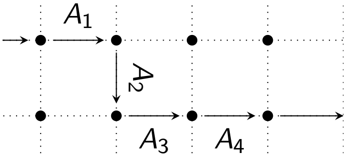

- Compositions of morphisms are called "paths."
- A path is a **complex** if compositions of 2 successive morphisms vanish:

$$A_{j+1} \circ A_j = 0.$$

- A complex is **exact** if $\mathrm{range}(A_j) = \ker(A_{j+1})$.
- A path is a **2-complex** if the compositions of 3 successive morphisms vanish: [Olver 1982]

$$A_{j+2} \circ A_{j+1} \circ A_j = 0.$$

Scalar ($\mathbb{R}$) and vector ($\mathbb{V}$) valued fields in 3D

On a bounded contractible domain $\Omega \subset \mathbb{R}^3$ with Lipschitz boundary, let $\mathcal{D}$ denote the space of infinitely differentiable functions compactly supported on Ω. The **de Rham complex**

$$0 \longrightarrow \mathcal{D} \xrightarrow{\text{grad}} \mathcal{D} \otimes \mathbb{V} \xrightarrow{\text{curl}} \mathcal{D} \otimes \mathbb{V} \xrightarrow{\text{div}} \mathcal{D} \longrightarrow \mathbb{R}$$

is exact.

<u>Notation:</u> $\mathbb{V} = \mathbb{R}^3$. Tensor product $X \otimes \mathbb{V}$ is identifiable with the Cartesian product $X \times X \times X$.

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is exact, and so is the Sobolev de Rham complex,

$$0 \longrightarrow \mathring{H}^1 \xrightarrow{\text{grad}} \mathring{H}(\text{curl}) \xrightarrow{\text{curl}} \mathring{H}(\text{div}) \xrightarrow{\text{div}} L_2 \longrightarrow \mathbb{R}$$

where

$$\mathring{H}(\text{curl}) = \overline{\mathcal{D} \otimes \mathbb{V}}^{\|\cdot\|_{H(\text{curl})}}, \qquad \mathring{H}(\text{div}) = \overline{\mathcal{D} \otimes \mathbb{V}}^{\|\cdot\|_{H(\text{div})}}$$

$$\|u\|^2_{H(\text{curl})} = \|u\|^2_{L_2} + \|\operatorname{curl} u\|^2_{L_2}, \qquad \|q\|^2_{H(\text{div})} = \|q\|^2_{L_2} + \|\operatorname{div} q\|^2_{L_2}.$$

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$\mathbb{S}$ and $\mathbb{T}$ valued matrix fields

$$\mathcal{D} \xrightarrow{\text{grad}} \mathcal{D} \otimes \mathbb{V} \xrightarrow{\text{curl}} \mathcal{D} \otimes \mathbb{V} \xrightarrow{\text{div}} \mathcal{D}$$

$$\mathcal{D} \otimes \mathbb{V} \qquad \mathcal{D} \otimes \mathbb{S} \qquad \mathcal{D} \otimes \mathbb{T} \qquad \mathcal{D} \otimes \mathbb{V}$$

Notation: 1 $\operatorname{sym} \tau = \frac{1}{2}(\tau + \tau^\top)$ for matrices τ. 2 $\mathbb{S} = \operatorname{sym} \mathbb{R}^{3\times 3}$, space of symmetric matrices.
3 $\operatorname{dev} \tau = \tau - \frac{1}{3}(\operatorname{tr} \tau) I$ for matrices τ, 4 $\mathbb{T} = \operatorname{dev} \mathbb{R}^{3\times 3}$ = space of traceless matrices.

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$$\mathcal{D} \xrightarrow{\text{grad}} \mathcal{D} \otimes \mathbb{V} \xrightarrow{\text{curl}} \mathcal{D} \otimes \mathbb{V} \xrightarrow{\text{div}} \mathcal{D}$$

$$\mathcal{D} \otimes \mathbb{V} \xrightarrow{\text{def}} \mathcal{D} \otimes \mathbb{S} \xrightarrow{\text{curl}} \mathcal{D} \otimes \mathbb{T} \xrightarrow{\text{div}} \mathcal{D} \otimes \mathbb{V}$$

- def $v \equiv \varepsilon(v) = \text{sym}\,\text{grad}\, v$, $\quad$ grad v = Jacobian matrix of vector field v,

Notation: (1) $\text{sym}\,\tau = \frac{1}{2}(\tau + \tau^\top)$ for matrices τ. (2) $\mathbb{S} = \text{sym}\,\mathbb{R}^{3\times 3}$, space of symmetric matrices.
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- $\operatorname{def} v \equiv \varepsilon(v) = \operatorname{sym} \operatorname{grad} v$, $\quad \operatorname{grad} v =$ Jacobian matrix of vector field v,
- curl and div on matrix-valued functions act row-wise.
- curl of a symmetric matrix is traceless.

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$\mathbb{S}$ and $\mathbb{T}$ valued matrix fields

$$
\begin{tikzcd}
\mathcal{D} \arrow[r, "\mathrm{grad}"] \arrow[d, "\mathrm{grad}"] & \mathcal{D} \otimes \mathbb{V} \arrow[r, "\mathrm{curl}"] \arrow[d, "\mathrm{def}"] & \mathcal{D} \otimes \mathbb{V} \arrow[r, "\mathrm{div}"] \arrow[d, "\frac{1}{2} \top {\color{red}\mathrm{dev}}\, \mathrm{grad}"] & \mathcal{D} \arrow[d, "\frac{1}{3}\, \mathrm{grad}"] \\
\mathcal{D} \otimes \mathbb{V} \arrow[r, "\mathrm{def}"] & \mathcal{D} \otimes \mathbb{S} \arrow[r, "\mathrm{curl}"] & \mathcal{D} \otimes \mathbb{T} \arrow[r, "\mathrm{div}"] & \mathcal{D} \otimes \mathbb{V}
\end{tikzcd}
$$

- Connect vertically.

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\mathcal{D} \otimes \mathbb{V} \arrow[r, "\mathrm{def}"] & \mathcal{D} \otimes \mathbb{S} \arrow[r, "\mathrm{curl}"] & \mathcal{D} \otimes \mathbb{T} \arrow[r, "\mathrm{div}"] & \mathcal{D} \otimes \mathbb{V}
\end{tikzcd}
$$

- The diagram commutes.

$$\mathrm{curl}\, \mathrm{def} = \frac{1}{2} \top \mathrm{dev}\, \mathrm{grad}\, \mathrm{curl}$$

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\mathcal{D}\otimes\mathbb{V} \arrow[r, "\text{def}"] & \mathcal{D}\otimes\mathbb{S} \arrow[r, "\text{curl}"] & \mathcal{D}\otimes\mathbb{T} \arrow[r, "\text{div}"] & \mathcal{D}\otimes\mathbb{V}
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$$

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$$\text{curl}\,\text{def} = \frac{1}{2}\top\,\text{dev}\,\text{grad}\,\text{curl}$$

$$\frac{1}{2}\text{div}\top\,\text{dev}\,\text{grad}\,u = \frac{1}{3}\text{grad}\,\text{div}$$

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Completing the picture

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\end{tikzcd}
$$

$$\mathrm{div}\,\mathrm{sym}\,\mathrm{curl}\,\top = \frac{1}{2}\,\mathrm{curl}\,\mathrm{div}\,.$$

Completing the picture

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\end{tikzcd}
$$

Note the symmetry in the diagram about the diagonal.

Completing the picture

$$
\begin{tikzcd}
\mathcal{D} \arrow[r, "\mathrm{grad}"] \arrow[d, "\mathrm{grad}"] & \mathcal{D} \otimes \mathbb{V} \arrow[r, "\mathrm{curl}"] \arrow[d, "\mathrm{def}"] & \mathcal{D} \otimes \mathbb{V} \arrow[r, "\mathrm{div}"] \arrow[d, "\frac{1}{2} \top \,\mathrm{dev}\,\mathrm{grad}"] & \mathcal{D} \arrow[d, "\frac{1}{3}\,\mathrm{grad}"] \\
\mathcal{D} \otimes \mathbb{V} \arrow[r, "\mathrm{def}"] \arrow[d, "\mathrm{curl}"] & \mathcal{D} \otimes {\color{red}\mathbb{S}} \arrow[r, "\mathrm{curl}"] \arrow[d, "\top \,\mathrm{curl}"] & \mathcal{D} \otimes {\color{red}\mathbb{T}} \arrow[r, "\mathrm{div}"] \arrow[d, "\mathrm{sym}\,\mathrm{curl}\, \top"] & \mathcal{D} \otimes \mathbb{V} \arrow[d, "\frac{1}{2}\,\mathrm{curl}"] \\
\mathcal{D} \otimes \mathbb{V} \arrow[r, "\frac{1}{2}\,\mathrm{dev}\,\mathrm{grad}"] \arrow[d, "\mathrm{div}"] & \mathcal{D} \otimes {\color{red}\mathbb{T}} \arrow[r, "\mathrm{sym}\,\mathrm{curl}"] \arrow[d, "\mathrm{div}\, \top"] & \mathcal{D} \otimes {\color{red}\mathbb{S}} \arrow[r, "\mathrm{div}"] \arrow[d, "\mathrm{div}"] & \mathcal{D} \otimes \mathbb{V} \arrow[d, "\mathrm{div}"] \\
\mathcal{D} \arrow[r, "\frac{1}{3}\,\mathrm{grad}"] & \mathcal{D} \otimes \mathbb{V} \arrow[r, "\frac{1}{2}\,\mathrm{curl}"] & \mathcal{D} \otimes \mathbb{V} \arrow[r, "\mathrm{div}"] & \mathcal{D}
\end{tikzcd}
$$

The four middle spaces extend to new Sobolev spaces of matrix fields.

New and old Sobolev spaces

Let

$$H^{-1} = (\mathring{H}^1)^*,$$
$$\mathring{H}^{-1} = (H^1)^*.$$

Define new Sobolev spaces of matrix fields:

$$\mathring{H}_{\mathrm{cc}} := \{g \in \mathring{H}^{-1} \otimes \mathbb{S} : \ \operatorname{curl} g \in \mathring{H}^{-1} \otimes \mathbb{T}, \ \operatorname{curl} \top \operatorname{curl} g \in \mathring{H}^{-1} \otimes \mathbb{S}\}.$$
$$\mathring{H}_{\mathrm{cd}} := \{\tau \in \mathring{H}^{-1} \otimes \mathbb{T} : \ \operatorname{div} \tau \in \mathring{H}^{-1} \otimes \mathbb{V}, \ \operatorname{sym}\operatorname{curl} \tau \in \mathring{H}^{-1} \otimes \mathbb{S}, \ \operatorname{curl}\operatorname{div} \tau \in \mathring{H}^{-1} \otimes \mathbb{V}\}.$$
$$\mathring{H}_{\mathrm{cd}\top} := \top \mathring{H}_{\mathrm{cd}}.$$
$$\mathring{H}_{\mathrm{dd}} := \{\sigma \in \mathring{H}^{-1} \otimes \mathbb{S} : \ \operatorname{div} \sigma \in \mathring{H}^{-1} \otimes \mathbb{V}, \ \operatorname{div}\operatorname{div} \sigma \in \mathring{H}^{-1}\}.$$

Define $H_{\mathrm{cc}}, H_{\mathrm{cd}}, H_{\mathrm{cd}\top}, H_{\mathrm{dd}}$ similarly, replacing $\mathring{H}^{-1}$ by H^{-1}.

Two de Rham complexes

Recall the standard Sobolev de Rham complex:

$$0 \longrightarrow \mathring{H}^1 \xrightarrow{\text{grad}} \mathring{H}(\text{curl}) \xrightarrow{\text{curl}} \mathring{H}(\text{div}) \xrightarrow{\text{div}} L_{2,\mathbb{R}}.$$

A version with lower regularity:

$$L_{2,\mathbb{R}} \xrightarrow{\text{grad}} \mathring{\tilde{H}}{}^{-1}_{\mathcal{RT}}(\text{curl}) \xrightarrow{\text{curl}} \mathring{\tilde{H}}{}^{-1}_{\mathcal{ND}}(\text{div}) \xrightarrow{\text{div}} \mathring{\tilde{H}}{}^{-1}_{\mathcal{P}_1}.$$

<u>Notation:</u>

$$L_{2,\mathbb{R}} = \{u \in L_2 : \textstyle\int_\Omega u = 0\},$$

$$\mathring{\tilde{H}}{}^{-1}_{\mathcal{RT}}(\text{curl}) = \{v \in \mathring{\tilde{H}}{}^{-1} : \text{curl}\, v \in \mathring{\tilde{H}}{}^{-1},\ v(r) = 0 \text{ for all } r \in \mathcal{RT}\}, \qquad \mathcal{RT} = \{a + bx : a \in \mathbb{V}, b \in \mathbb{R}\},$$

$$\mathring{\tilde{H}}{}^{-1}_{\mathcal{ND}}(\text{div}) = \{q \in \mathring{\tilde{H}}{}^{-1} : \text{div}\, q \in \mathring{\tilde{H}}{}^{-1},\ q(r) = 0 \text{ for all } r \in \mathcal{ND}\}, \qquad \mathcal{ND} = \{a + d \times x : a, d \in \mathbb{V}\},$$

$$\mathring{\tilde{H}}{}^{-1}_{\mathcal{P}_1} = \{w \in \mathring{\tilde{H}}{}^{-1} : w(p) = 0 \text{ for all } p \in \mathcal{P}_1\}, \qquad \mathcal{P}_k = \text{polynomials of degree } \le k.$$

Two de Rham complexes

Recall the standard Sobolev de Rham complex:

$$0 \longrightarrow \mathring{H}^1 \xrightarrow{\text{grad}} \mathring{H}(\text{curl}) \xrightarrow{\text{curl}} \mathring{H}(\text{div}) \xrightarrow{\text{div}} L_{2,\mathbb{R}}.$$

A version with lower regularity:

$$L_{2,\mathbb{R}} \xrightarrow{\text{grad}} \tilde{H}^{-1}_{\mathcal{RT}}(\text{curl}) \xrightarrow{\text{curl}} \tilde{H}^{-1}_{\mathcal{ND}}(\text{div}) \xrightarrow{\text{div}} \tilde{H}^{-1}_{\mathcal{P}_1}.$$

(*Why remove* $\mathcal{RT}$? $\quad u \in L_{2,\mathbb{R}} \implies (\text{grad}\, u)(a + bx) = -u(\text{div}(a + bx)) = -b \int_\Omega u = 0.$)

Notation:

$$L_{2,\mathbb{R}} = \{u \in L_2 : \int_\Omega u = 0\},$$

$$\tilde{H}^{-1}_{\mathcal{RT}}(\text{curl}) = \{v \in \tilde{H}^{-1} : \text{curl}\, v \in \tilde{H}^{-1},\ v(r) = 0 \text{ for all } r \in \mathcal{RT}\}, \qquad \mathcal{RT} = \{a + bx : a \in \mathbb{V}, b \in \mathbb{R}\},$$

$$\tilde{H}^{-1}_{\mathcal{ND}}(\text{div}) = \{q \in \tilde{H}^{-1} : \text{div}\, q \in \tilde{H}^{-1},\ q(r) = 0 \text{ for all } r \in \mathcal{ND}\}, \qquad \mathcal{ND} = \{a + d \times x : a, d \in \mathbb{V}\},$$

$$\tilde{H}^{-1}_{\mathcal{P}_1} = \{w \in \tilde{H}^{-1} : w(p) = 0 \text{ for all } p \in \mathcal{P}_1\}, \qquad \mathcal{P}_k = \text{polynomials of degree } \le k.$$

The diagram with Sobolev spaces

$$
\begin{tikzcd}
\mathring{H}(\mathrm{grad}) \arrow[r, "\mathrm{grad}"] \arrow[d, "\mathrm{grad}"] & \mathring{H}(\mathrm{curl}) \arrow[r, "\mathrm{curl}"] \arrow[d, "\mathrm{def}"] & \mathring{H}(\mathrm{div}) \arrow[r, "\mathrm{div}"] \arrow[d, "\frac{1}{2}\top\,\mathrm{dev}\,\mathrm{grad}"] & L_{2,\mathbb{R}} \arrow[d, "\frac{1}{3}\,\mathrm{grad}"] \\
\mathring{H}(\mathrm{curl}) \arrow[r, "\mathrm{def}"] \arrow[d, "\mathrm{curl}"] & \mathring{\tilde{H}}_{\mathrm{cc}} \arrow[r, "\mathrm{curl}"] \arrow[d, "\top\,\mathrm{curl}"] & \mathring{\tilde{H}}_{\mathrm{cd}} \arrow[r, "\mathrm{div}"] \arrow[d, "\mathrm{sym}\,\mathrm{curl}\,\top"] & \mathring{\tilde{H}}^{-1}_{\mathcal{R}\mathcal{T}}(\mathrm{curl}) \arrow[d, "\frac{1}{2}\,\mathrm{curl}"] \\
\mathring{H}(\mathrm{div}) \arrow[r, "\frac{1}{2}\,\mathrm{dev}\,\mathrm{grad}"] \arrow[d, "\mathrm{div}"] & \mathring{\tilde{H}}_{\mathrm{cd}\top} \arrow[r, "\mathrm{sym}\,\mathrm{curl}"] \arrow[d, "\mathrm{div}\,\top"] & \mathring{\tilde{H}}_{\mathrm{dd}} \arrow[r, "\mathrm{div}"] \arrow[d, "\mathrm{div}"] & \mathring{\tilde{H}}^{-1}_{\mathcal{N}\mathcal{D}}(\mathrm{div}) \arrow[d, "\mathrm{div}"] \\
L_{2,\mathbb{R}} \arrow[r, "\frac{1}{3}\,\mathrm{grad}"] & \mathring{\tilde{H}}^{-1}_{\mathcal{R}\mathcal{T}}(\mathrm{curl}) \arrow[r, "\frac{1}{2}\,\mathrm{curl}"] & \mathring{\tilde{H}}^{-1}_{\mathcal{N}\mathcal{D}}(\mathrm{div}) \arrow[r, "\mathrm{div}"] & \mathring{\tilde{H}}^{-1}_{\mathcal{P}_1}
\end{tikzcd}
$$

Theorem: The diagram commutes and every differential operator in it is continuous (with respect to the norms of the indicated domains and codomains) and has closed range.

2-complexes

$$
\begin{tikzcd}
\mathring{H}(\mathrm{grad}) \arrow[r, "\mathrm{grad}"] \arrow[d, "\mathrm{grad}"] & \mathring{H}(\mathrm{curl}) \arrow[r, "\mathrm{curl}"] \arrow[d, "\mathrm{def}"] & \mathring{H}(\mathrm{div}) \arrow[r, "\mathrm{div}"] \arrow[d, "\frac{1}{2}\top\,\mathrm{dev}\,\mathrm{grad}"] & L_{2,\mathbb{R}} \arrow[d, "\frac{1}{3}\,\mathrm{grad}"] \\
\mathring{H}(\mathrm{curl}) \arrow[r, "\mathrm{def}"] \arrow[d, "\mathrm{curl}"] & \mathring{\tilde{H}}_{\mathrm{cc}} \arrow[r, "\mathrm{curl}"] \arrow[d, "\top\,\mathrm{curl}"] & \mathring{\tilde{H}}_{\mathrm{cd}} \arrow[r, "\mathrm{div}"] \arrow[d, "\mathrm{sym}\,\mathrm{curl}\,\top"] & \mathring{\tilde{H}}^{-1}_{\mathcal{RT}}(\mathrm{curl}) \arrow[d, "\frac{1}{2}\,\mathrm{curl}"] \\
\mathring{H}(\mathrm{div}) \arrow[r, "\frac{1}{2}\,\mathrm{dev}\,\mathrm{grad}"] \arrow[d, "\mathrm{div}"] & \mathring{\tilde{H}}_{\mathrm{cd}\top} \arrow[r, "\mathrm{sym}\,\mathrm{curl}"] \arrow[d, "\mathrm{div}\,\top"] & \mathring{\tilde{H}}_{\mathrm{dd}} \arrow[r, "\mathrm{div}"] \arrow[d, "\mathrm{div}"] & \mathring{\tilde{H}}^{-1}_{\mathcal{ND}}(\mathrm{div}) \arrow[d, "\mathrm{div}"] \\
L_{2,\mathbb{R}} \arrow[r, "\frac{1}{3}\,\mathrm{grad}"] & \mathring{\tilde{H}}^{-1}_{\mathcal{RT}}(\mathrm{curl}) \arrow[r, "\frac{1}{2}\,\mathrm{curl}"] & \mathring{\tilde{H}}^{-1}_{\mathcal{ND}}(\mathrm{div}) \arrow[r, "\mathrm{div}"] & \mathring{\tilde{H}}^{-1}_{\mathcal{P}_1}
\end{tikzcd}
$$

Theorem: All paths in the diagram are 2-complexes.

2-complexes

$$
\begin{tikzcd}
\mathring{H}(\mathrm{grad}) \arrow[r, "\mathrm{grad}"] \arrow[d, "\mathrm{grad}"] & \mathring{H}(\mathrm{curl}) \arrow[r, "\mathrm{curl}"] \arrow[d, "\mathrm{def}"] & \mathring{H}(\mathrm{div}) \arrow[r, "\mathrm{div}"] \arrow[d, "\frac{1}{2}\top\,\mathrm{dev}\,\mathrm{grad}"] & L_{2,\mathbb{R}} \arrow[d, "\frac{1}{3}\,\mathrm{grad}"] \\
\mathring{H}(\mathrm{curl}) \arrow[r, "\mathrm{def}"] \arrow[d, "\mathrm{curl}"] & \mathring{\tilde{H}}_{\mathrm{cc}} \arrow[r, "\mathrm{curl}"] \arrow[d, "\top\,\mathrm{curl}"] & \mathring{\tilde{H}}_{\mathrm{cd}} \arrow[r, "\mathrm{div}"] \arrow[d, "\mathrm{sym}\,\mathrm{curl}\,\top"] & \mathring{\tilde{H}}^{-1}_{\mathcal{RT}}(\mathrm{curl}) \arrow[d, "\frac{1}{2}\,\mathrm{curl}"] \\
\mathring{H}(\mathrm{div}) \arrow[r, "\frac{1}{2}\,\mathrm{dev}\,\mathrm{grad}"] \arrow[d, "\mathrm{div}"] & \mathring{\tilde{H}}_{\mathrm{cd}\top} \arrow[r, "\mathrm{sym}\,\mathrm{curl}"] \arrow[d, "\mathrm{div}\,\top"] & \mathring{\tilde{H}}_{\mathrm{dd}} \arrow[r, "\mathrm{div}"] \arrow[d, "\mathrm{div}"] & \mathring{\tilde{H}}^{-1}_{\mathcal{ND}}(\mathrm{div}) \arrow[d, "\mathrm{div}"] \\
L_{2,\mathbb{R}} \arrow[r, "\frac{1}{3}\,\mathrm{grad}"] & \mathring{\tilde{H}}^{-1}_{\mathcal{RT}}(\mathrm{curl}) \arrow[r, "\frac{1}{2}\,\mathrm{curl}"] & \mathring{\tilde{H}}^{-1}_{\mathcal{ND}}(\mathrm{div}) \arrow[r, "\mathrm{div}"] & \mathring{\tilde{H}}^{-1}_{\mathcal{P}_1}
\end{tikzcd}
$$

E.g., $\mathrm{curl} \circ \mathrm{def} \circ \mathrm{grad}\, w = \frac{1}{2} \top\, \mathrm{dev}\,\mathrm{grad}\,\mathrm{curl}\,\mathrm{grad}\, w = 0.$

2-complexes

$$
\begin{tikzcd}
\mathring{H}(\mathrm{grad}) \arrow[r, "\mathrm{grad}"] \arrow[d, "\mathrm{grad}"] & \mathring{H}(\mathrm{curl}) \arrow[r, "\mathrm{curl}"] \arrow[d, "\mathrm{def}"] & \mathring{H}(\mathrm{div}) \arrow[r, "\mathrm{div}"] \arrow[d, "\frac{1}{2} \top \mathrm{dev}\,\mathrm{grad}"] & L_{2,\mathbb{R}} \arrow[d, "\frac{1}{3} \mathrm{grad}"] \\
\mathring{H}(\mathrm{curl}) \arrow[r, "\mathrm{def}"] \arrow[d, "\mathrm{curl}"] & \mathring{\tilde{H}}_{\mathrm{cc}} \arrow[r, "\mathrm{curl}"] \arrow[d, "\top \mathrm{curl}"] & \mathring{\tilde{H}}_{\mathrm{cd}} \arrow[r, "\mathrm{div}"] \arrow[d, "\mathrm{sym}\,\mathrm{curl} \top"] & \mathring{\tilde{H}}^{-1}_{\mathcal{RT}}(\mathrm{curl}) \arrow[d, "\frac{1}{2} \mathrm{curl}"] \\
\mathring{H}(\mathrm{div}) \arrow[r, "\frac{1}{2} \mathrm{dev}\,\mathrm{grad}"] \arrow[d, "\mathrm{div}"] & \mathring{\tilde{H}}_{\mathrm{cd}\top} \arrow[r, "\mathrm{sym}\,\mathrm{curl}"] \arrow[d, "\mathrm{div} \top"] & \mathring{\tilde{H}}_{\mathrm{dd}} \arrow[r, "\mathrm{div}"] \arrow[d, "\mathrm{div}"] & \mathring{\tilde{H}}^{-1}_{\mathcal{ND}}(\mathrm{div}) \arrow[d, "\mathrm{div}"] \\
L_{2,\mathbb{R}} \arrow[r, "\frac{1}{3} \mathrm{grad}"] & \mathring{\tilde{H}}^{-1}_{\mathcal{RT}}(\mathrm{curl}) \arrow[r, "\frac{1}{2} \mathrm{curl}"] & \mathring{\tilde{H}}^{-1}_{\mathcal{ND}}(\mathrm{div}) \arrow[r, "\mathrm{div}"] & \mathring{\tilde{H}}^{-1}_{\mathcal{P}_1}
\end{tikzcd}
$$

E.g., $\mathrm{div} \circ \mathrm{div} \circ \mathrm{sym}\,\mathrm{curl}\,\tau = \frac{1}{2}\, \mathrm{div}\,\mathrm{curl}\,\mathrm{div} \top \tau = 0.$

Diagonal second-order operators

$$
\begin{tikzcd}
\mathring{H}(\mathrm{grad}) \arrow[r, "\mathrm{grad}"] \arrow[d, "\mathrm{grad}"] \arrow[dr, "\mathrm{hess}"] & \mathring{H}(\mathrm{curl}) \arrow[r, "\mathrm{curl}"] \arrow[d, "\mathrm{def}"] \arrow[dr, "\mathrm{curl\,def}"] & \mathring{H}(\mathrm{div}) \arrow[r, "\mathrm{div}"] \arrow[d, "\frac{1}{2}\top\,\mathrm{dev\,grad}"] \arrow[dr, "\frac{1}{3}\,\mathrm{grad\,div}"] & L_{2,\mathbb{R}} \arrow[d, "\frac{1}{3}\,\mathrm{grad}"] \\
\mathring{H}(\mathrm{curl}) \arrow[r, "\mathrm{def}"] \arrow[d, "\mathrm{curl}"] & \mathring{\tilde{H}}_{\mathrm{cc}} \arrow[r, "\mathrm{curl}"] \arrow[d, "\top\,\mathrm{curl}"] \arrow[dr, "\mathrm{inc}"] & \mathring{\tilde{H}}_{\mathrm{cd}} \arrow[r, "\mathrm{div}"] \arrow[d, "\mathrm{sym\,curl}\,\top"] \arrow[dr, "\frac{1}{2}\,\mathrm{curl\,div}"] & \mathring{\tilde{H}}^{-1}_{\mathcal{RT}}(\mathrm{curl}) \arrow[d, "\frac{1}{2}\,\mathrm{curl}"] \\
\mathring{H}(\mathrm{div}) \arrow[r, "\frac{1}{2}\,\mathrm{dev\,grad}"] \arrow[d, "\mathrm{div}"] & \mathring{\tilde{H}}_{\mathrm{cd}\top} \arrow[r, "\mathrm{sym\,curl}"] \arrow[d, "\mathrm{div}\,\top"] & \mathring{\tilde{H}}_{\mathrm{dd}} \arrow[r, "\mathrm{div}"] \arrow[d, "\mathrm{div}"] \arrow[dr, "\mathrm{div\,div}"] & \mathring{\tilde{H}}^{-1}_{\mathcal{ND}}(\mathrm{div}) \arrow[d, "\mathrm{div}"] \\
L_{2,\mathbb{R}} \arrow[r, "\frac{1}{3}\,\mathrm{grad}"] & \mathring{\tilde{H}}^{-1}_{\mathcal{RT}}(\mathrm{curl}) \arrow[r, "\frac{1}{2}\,\mathrm{curl}"] & \mathring{\tilde{H}}^{-1}_{\mathcal{ND}}(\mathrm{div}) \arrow[r, "\mathrm{div}"] & \mathring{\tilde{H}}^{-1}_{\mathcal{P}_1}
\end{tikzcd}
$$

Diagonal second-order operators

$$
\begin{tikzcd}
\mathring{H}(\mathrm{grad}) \arrow[r, "\mathrm{grad}"] \arrow[d, "\mathrm{grad}"] \arrow[dr, "\mathrm{hess}"] & \mathring{H}(\mathrm{curl}) \arrow[r, "\mathrm{curl}"] \arrow[d, "\mathrm{def}"] \arrow[dr, "\mathrm{curl\,def}"] & \mathring{H}(\mathrm{div}) \arrow[r, "\mathrm{div}"] \arrow[d, "\frac{1}{2}\top\,\mathrm{dev\,grad}"] \arrow[dr, "\frac{1}{3}\mathrm{grad\,div}"] & L_{2,\mathbb{R}} \arrow[d, "\frac{1}{3}\mathrm{grad}"] \\
\mathring{H}(\mathrm{curl}) \arrow[r, "\mathrm{def}"] \arrow[d, "\mathrm{curl}"] & \tilde{\mathring{H}}_{\mathrm{cc}} \arrow[r, "\mathrm{curl}"] \arrow[d, "\top\,\mathrm{curl}"] \arrow[dr, "\mathrm{inc}"] & \tilde{\mathring{H}}_{\mathrm{cd}} \arrow[r, "\mathrm{div}"] \arrow[d, "\mathrm{sym\,curl}\,\top"] \arrow[dr, "\frac{1}{2}\mathrm{curl\,div}"] & \tilde{\mathring{H}}^{-1}_{\mathcal{RT}}(\mathrm{curl}) \arrow[d, "\frac{1}{2}\mathrm{curl}"] \\
\mathring{H}(\mathrm{div}) \arrow[r, "\frac{1}{2}\mathrm{dev\,grad}"] \arrow[d, "\mathrm{div}"] & \tilde{\mathring{H}}_{\mathrm{cd}\top} \arrow[r, "\mathrm{sym\,curl}"] \arrow[d, "\mathrm{div}\,\top"] & \tilde{\mathring{H}}_{\mathrm{dd}} \arrow[r, "\mathrm{div}"] \arrow[d, "\mathrm{div}"] \arrow[dr, "\mathrm{div\,div}"] & \tilde{\mathring{H}}^{-1}_{\mathcal{ND}}(\mathrm{div}) \arrow[d, "\mathrm{div}"] \\
L_{2,\mathbb{R}} \arrow[r, "\frac{1}{3}\mathrm{grad}"] & \tilde{\mathring{H}}^{-1}_{\mathcal{RT}}(\mathrm{curl}) \arrow[r, "\frac{1}{2}\mathrm{curl}"] & \tilde{\mathring{H}}^{-1}_{\mathcal{ND}}(\mathrm{div}) \arrow[r, "\mathrm{div}"] & \tilde{\mathring{H}}^{-1}_{\mathcal{P}_1}
\end{tikzcd}
$$

Operators that appeared

- in MCS: curl div

Diagonal second-order operators

$$
\begin{tikzcd}
\mathring{H}(\mathrm{grad}) \arrow[r, "\mathrm{grad}"] \arrow[d, "\mathrm{grad}"] \arrow[dr, "\mathrm{hess}"] & \mathring{H}(\mathrm{curl}) \arrow[r, "\mathrm{curl}"] \arrow[d, "\mathrm{def}"] \arrow[dr, "\mathrm{curl\,def}"] & \mathring{H}(\mathrm{div}) \arrow[r, "\mathrm{div}"] \arrow[d, "\frac{1}{2}\top\mathrm{dev\,grad}"] \arrow[dr, "\frac{1}{3}\mathrm{grad\,div}"] & L_{2,\mathbb{R}} \arrow[d, "\frac{1}{3}\mathrm{grad}"] \\
\mathring{H}(\mathrm{curl}) \arrow[r, "\mathrm{def}"] \arrow[d, "\mathrm{curl}"] & \mathring{\tilde{H}}_{\mathrm{cc}} \arrow[r, "\mathrm{curl}"] \arrow[d, "\top\mathrm{curl}"] \arrow[dr, "\mathrm{inc}"] & \mathring{\tilde{H}}_{\mathrm{cd}} \arrow[r, "\mathrm{div}"] \arrow[d, "\mathrm{sym\,curl}\top"] \arrow[dr, "\frac{1}{2}\mathrm{curl\,div}"] & \mathring{\tilde{H}}^{-1}_{\mathcal{RT}}(\mathrm{curl}) \arrow[d, "\frac{1}{2}\mathrm{curl}"] \\
\mathring{H}(\mathrm{div}) \arrow[r, "\frac{1}{2}\mathrm{dev\,grad}"] \arrow[d, "\mathrm{div}"] & \mathring{\tilde{H}}_{\mathrm{cd}\top} \arrow[r, "\mathrm{sym\,curl}"] \arrow[d, "\mathrm{div}\top"] & \mathring{\tilde{H}}_{\mathrm{dd}} \arrow[r, "\mathrm{div}"] \arrow[d, "\mathrm{div}"] \arrow[dr, "\mathrm{div\,div}"] & \mathring{\tilde{H}}^{-1}_{\mathcal{ND}}(\mathrm{div}) \arrow[d, "\mathrm{div}"] \\
L_{2,\mathbb{R}} \arrow[r, "\frac{1}{3}\mathrm{grad}"] & \mathring{\tilde{H}}^{-1}_{\mathcal{RT}}(\mathrm{curl}) \arrow[r, "\frac{1}{2}\mathrm{curl}"] & \mathring{\tilde{H}}^{-1}_{\mathcal{ND}}(\mathrm{div}) \arrow[r, "\mathrm{div}"] & \mathring{\tilde{H}}^{-1}_{\mathcal{P}_1}
\end{tikzcd}
$$

Operators that appeared

- in MCS: $\mathrm{curl\,div}$
- in curvature: $\mathrm{inc} = \mathrm{curl}\top\mathrm{curl}$

Diagonal second-order operators

$$
\begin{tikzcd}
\mathring{H}(\mathrm{grad}) \arrow[r, "\mathrm{grad}"] \arrow[d, "\mathrm{grad}"] \arrow[dr, "\mathrm{hess}"] & \mathring{H}(\mathrm{curl}) \arrow[r, "\mathrm{curl}"] \arrow[d, "\mathrm{def}"] \arrow[dr, "\mathrm{curl\,def}"] & \mathring{H}(\mathrm{div}) \arrow[r, "\mathrm{div}"] \arrow[d, "\frac{1}{2}\top\,\mathrm{dev\,grad}"] \arrow[dr, "\frac{1}{3}\mathrm{grad\,div}"] & L_{2,\mathbb{R}} \arrow[d, "\frac{1}{3}\mathrm{grad}"] \\
\mathring{H}(\mathrm{curl}) \arrow[r, "\mathrm{def}"] \arrow[d, "\mathrm{curl}"] & \mathring{\tilde{H}}_{\mathrm{cc}} \arrow[r, "\mathrm{curl}"] \arrow[d, "\top\,\mathrm{curl}"] \arrow[dr, "\mathrm{inc}"] & \mathring{\tilde{H}}_{\mathrm{cd}} \arrow[r, "\mathrm{div}"] \arrow[d, "\mathrm{sym\,curl}\,\top"] \arrow[dr, "\frac{1}{2}\mathrm{curl\,div}"] & \mathring{\tilde{H}}^{-1}_{\mathcal{RT}}(\mathrm{curl}) \arrow[d, "\frac{1}{2}\mathrm{curl}"] \\
\mathring{H}(\mathrm{div}) \arrow[r, "\frac{1}{2}\mathrm{dev\,grad}"] \arrow[d, "\mathrm{div}"] & \mathring{\tilde{H}}_{\mathrm{cd}\top} \arrow[r, "\mathrm{sym\,curl}"] \arrow[d, "\mathrm{div}\,\top"] & \mathring{\tilde{H}}_{\mathrm{dd}} \arrow[r, "\mathrm{div}"] \arrow[d, "\mathrm{div}"] \arrow[dr, "\mathrm{div\,div}"] & \mathring{\tilde{H}}^{-1}_{\mathcal{ND}}(\mathrm{div}) \arrow[d, "\mathrm{div}"] \\
L_{2,\mathbb{R}} \arrow[r, "\frac{1}{3}\mathrm{grad}"] & \mathring{\tilde{H}}^{-1}_{\mathcal{RT}}(\mathrm{curl}) \arrow[r, "\frac{1}{2}\mathrm{curl}"] & \mathring{\tilde{H}}^{-1}_{\mathcal{ND}}(\mathrm{div}) \arrow[r, "\mathrm{div}"] & \mathring{\tilde{H}}^{-1}_{\mathcal{P}_1}
\end{tikzcd}
$$

Operators that appeared

- in MCS: curl div
- in curvature: inc $=$ curl $\top$ curl
- in TDNNS: div div

Complexes

Theorem:
The following paths are exact complexes:
The hessian complex

$$
\begin{tikzcd}
\mathring{H}(\mathrm{grad}) \arrow[r, "\mathrm{grad}"] \arrow[d, "\mathrm{grad}"] \arrow[dr, "\mathrm{hess}"] & \mathring{H}(\mathrm{curl}) \arrow[r, "\mathrm{curl}"] \arrow[d, "\mathrm{def}"] \arrow[dr, "\mathrm{curl\,def}"] & \mathring{H}(\mathrm{div}) \arrow[r, "\mathrm{div}"] \arrow[d, "\frac{1}{2}\top\,\mathrm{dev\,grad}"] \arrow[dr, "\frac{1}{3}\,\mathrm{grad\,div}"] & L_{2,\mathbb{R}} \arrow[d, "\frac{1}{3}\,\mathrm{grad}"] \\
\mathring{H}(\mathrm{curl}) \arrow[r, "\mathrm{def}"] \arrow[d, "\mathrm{curl}"] & \mathring{\tilde{H}}_{\mathrm{cc}} \arrow[r, "\mathrm{curl}"] \arrow[d, "\top\,\mathrm{curl}"] \arrow[dr, "\mathrm{inc}"] & \mathring{\tilde{H}}_{\mathrm{cd}} \arrow[r, "\mathrm{div}"] \arrow[d, "\mathrm{sym\,curl}\,\top"] \arrow[dr, "\frac{1}{2}\,\mathrm{curl\,div}"] & \mathring{\tilde{H}}^{-1}_{\mathcal{RT}}(\mathrm{curl}) \arrow[d, "\frac{1}{2}\,\mathrm{curl}"] \\
\mathring{H}(\mathrm{div}) \arrow[r, "\frac{1}{2}\,\mathrm{dev\,grad}"] \arrow[d, "\mathrm{div}"] & \mathring{\tilde{H}}_{\mathrm{cd}\top} \arrow[r, "\mathrm{sym\,curl}"] \arrow[d, "\mathrm{div}\,\top"] & \mathring{\tilde{H}}_{\mathrm{dd}} \arrow[r, "\mathrm{div}"] \arrow[d, "\mathrm{div}"] \arrow[dr, "\mathrm{div\,div}"] & \mathring{\tilde{H}}^{-1}_{\mathcal{ND}}(\mathrm{div}) \arrow[d, "\mathrm{div}"] \\
L_{2,\mathbb{R}} \arrow[r, "\frac{1}{3}\,\mathrm{grad}"] & \mathring{\tilde{H}}^{-1}_{\mathcal{RT}}(\mathrm{curl}) \arrow[r, "\frac{1}{2}\,\mathrm{curl}"] & \mathring{\tilde{H}}^{-1}_{\mathcal{ND}}(\mathrm{div}) \arrow[r, "\mathrm{div}"] & \mathring{\tilde{H}}^{-1}_{\mathcal{P}_1}
\end{tikzcd}
$$

Complexes

Theorem:
The following paths are exact complexes:

The hessian complex

The elasticity complex

$$
\begin{tikzcd}
\mathring{H}(\mathrm{grad}) \arrow[r, "\mathrm{grad}"] \arrow[d, "\mathrm{grad}"] \arrow[dr, "\mathrm{hess}"] & \mathring{H}(\mathrm{curl}) \arrow[r, "\mathrm{curl}"] \arrow[d, "\mathrm{def}"] \arrow[dr, "\mathrm{curl\,def}"] & \mathring{H}(\mathrm{div}) \arrow[r, "\mathrm{div}"] \arrow[d, "\frac{1}{2}\top\,\mathrm{dev\,grad}"] \arrow[dr, "\frac{1}{3}\mathrm{grad\,div}"] & L_{2,\mathbb{R}} \arrow[d, "\frac{1}{3}\mathrm{grad}"] \\
\mathring{H}(\mathrm{curl}) \arrow[r, "\mathrm{def}"] \arrow[d, "\mathrm{curl}"] & \mathring{\widetilde{H}}_{\mathrm{cc}} \arrow[r, "\mathrm{curl}"] \arrow[d, "\top\mathrm{curl}"] \arrow[dr, "\mathrm{inc}"] & \mathring{\widetilde{H}}_{\mathrm{cd}} \arrow[r, "\mathrm{div}"] \arrow[d, "\mathrm{sym\,curl}\top"] \arrow[dr, "\frac{1}{2}\mathrm{curl\,div}"] & \mathring{\widetilde{H}}^{-1}_{\mathcal{RT}}(\mathrm{curl}) \arrow[d, "\frac{1}{2}\mathrm{curl}"] \\
\mathring{H}(\mathrm{div}) \arrow[r, "\frac{1}{2}\mathrm{dev\,grad}"] \arrow[d, "\mathrm{div}"] & \mathring{\widetilde{H}}_{\mathrm{cd}\top} \arrow[r, "\mathrm{sym\,curl}"] \arrow[d, "\mathrm{div}\top"] & \mathring{\widetilde{H}}_{\mathrm{dd}} \arrow[r, "\mathrm{div}"] \arrow[d, "\mathrm{div}"] \arrow[dr, "\mathrm{div\,div}"] & \mathring{\widetilde{H}}^{-1}_{\mathcal{ND}}(\mathrm{div}) \arrow[d, "\mathrm{div}"] \\
L_{2,\mathbb{R}} \arrow[r, "\frac{1}{3}\mathrm{grad}"] & \mathring{\widetilde{H}}^{-1}_{\mathcal{RT}}(\mathrm{curl}) \arrow[r, "\frac{1}{2}\mathrm{curl}"] & \mathring{\widetilde{H}}^{-1}_{\mathcal{ND}}(\mathrm{div}) \arrow[r, "\mathrm{div}"] & \mathring{\widetilde{H}}^{-1}_{\mathcal{P}_1}
\end{tikzcd}
$$

Complexes

Theorem:
The following paths are exact complexes:

The hessian complex

The elasticity complex

The div div complex

$$
\begin{tikzcd}
\mathring{H}(\mathrm{grad}) \arrow[r, "\mathrm{grad}"] \arrow[d, "\mathrm{grad}"] \arrow[dr, "\mathrm{hess}"] & \mathring{H}(\mathrm{curl}) \arrow[r, "\mathrm{curl}"] \arrow[d, "\mathrm{def}"] \arrow[dr, "\mathrm{curl\,def}"] & \mathring{H}(\mathrm{div}) \arrow[r, "\mathrm{div}"] \arrow[d, "\frac{1}{2}\top\,\mathrm{dev\,grad}"] \arrow[dr, "\frac{1}{3}\mathrm{grad\,div}"] & L_{2,\mathbb{R}} \arrow[d, "\frac{1}{3}\mathrm{grad}"] \\
\mathring{H}(\mathrm{curl}) \arrow[r, "\mathrm{def}"] \arrow[d, "\mathrm{curl}"] & \mathring{\tilde{H}}_{\mathrm{cc}} \arrow[r, "\mathrm{curl}"] \arrow[d, "\top\,\mathrm{curl}"] \arrow[dr, "\mathrm{inc}"] & \mathring{\tilde{H}}_{\mathrm{cd}} \arrow[r, "\mathrm{div}"] \arrow[d, "\mathrm{sym\,curl}\,\top"] \arrow[dr, "\frac{1}{2}\mathrm{curl\,div}"] & \mathring{\tilde{H}}^{-1}_{\mathcal{RT}}(\mathrm{curl}) \arrow[d, "\frac{1}{2}\mathrm{curl}"] \\
\mathring{H}(\mathrm{div}) \arrow[r, "\frac{1}{2}\mathrm{dev\,grad}"] \arrow[d, "\mathrm{div}"] & \mathring{\tilde{H}}_{\mathrm{cd}\top} \arrow[r, "\mathrm{sym\,curl}"] \arrow[d, "\mathrm{div}\,\top"] & \mathring{\tilde{H}}_{\mathrm{dd}} \arrow[r, "\mathrm{div}"] \arrow[d, "\mathrm{div}"] \arrow[dr, "\mathrm{div\,div}"] & \mathring{\tilde{H}}^{-1}_{\mathcal{ND}}(\mathrm{div}) \arrow[d, "\mathrm{div}"] \\
L_{2,\mathbb{R}} \arrow[r, "\frac{1}{3}\mathrm{grad}"] & \mathring{\tilde{H}}^{-1}_{\mathcal{RT}}(\mathrm{curl}) \arrow[r, "\frac{1}{2}\mathrm{curl}"] & \mathring{\tilde{H}}^{-1}_{\mathcal{ND}}(\mathrm{div}) \arrow[r, "\mathrm{div}"] & \mathring{\tilde{H}}^{-1}_{\mathcal{P}_1}
\end{tikzcd}
$$

Outline

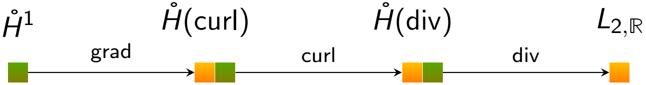
Generalized Bogovskiĭ operators

- A classical regular right inverse of $\operatorname{div}(\cdot)$ was developed by [Bogovskiĭ 1979]. It was generalized to r-forms in n-dimensions by [Costabel+McIntosh 2010]. In 3D,

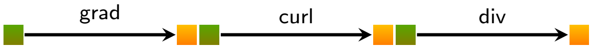
$$\mathring{H}^1 \underset{T_{\mathrm{g}}}{\overset{\text{grad}}{\rightleftarrows}} \mathring{H}(\text{curl}) \underset{T_{\mathrm{c}}}{\overset{\text{curl}}{\rightleftarrows}} \mathring{H}(\text{div}) \underset{T_{\mathrm{d}}}{\overset{\text{div}}{\rightleftarrows}} L_{2,\mathbb{R}},$$

  where T_{g}, T_{c}, and T_{d} map into more regular subspaces with $\mathring{H}^1$ components.

- Consequently, the spaces can be split into smooth and rough parts:



Regular decompositions of the spaces



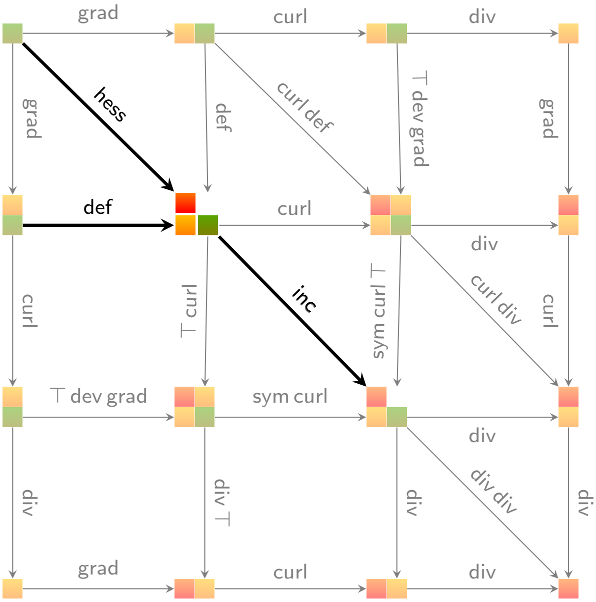
$\mathring{H}^1$	(the good)
L_2	(the bad)

The standard case

There are continuous linear maps
$S_{\mathrm{c}} : \mathring{H}(\mathrm{curl}) \to \mathring{H}^1$ and
$S_{\mathrm{d}} : \mathring{H}(\mathrm{div}) \to \mathring{H}^1 \otimes \mathbb{V}$ such that

$$\forall u \in \mathring{H}(\mathrm{curl}) :$$
$$u = \mathrm{grad}(S_{\mathrm{c}}\, u) + T_{\mathrm{c}}\, \mathrm{curl}\, u,$$
$$\forall q \in \mathring{H}(\mathrm{div}) :$$
$$q = \mathrm{curl}(S_{\mathrm{d}}\, q) + T_{\mathrm{d}}\, \mathrm{div}\, q.$$

Regular decompositions of the spaces



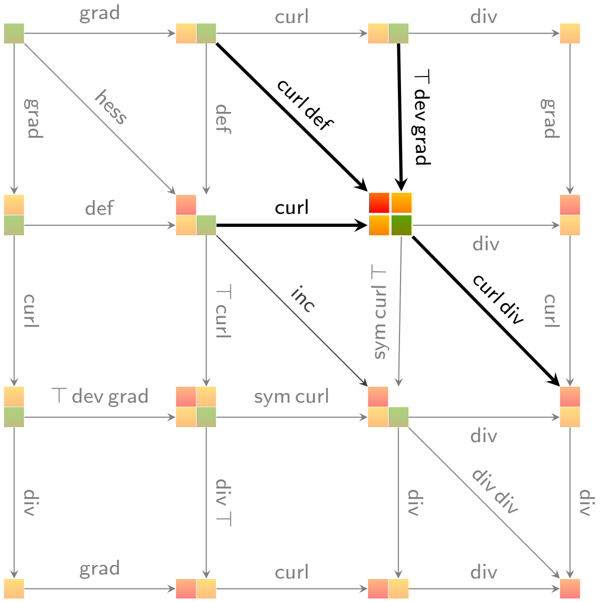
■	$\mathring{H}^1$	(the good)
■	L_2	(the bad)
■	$\tilde{\mathring{H}}^{-1}$	(the ugly)

The $\tilde{\mathring{H}}_{\mathrm{cc}}$ case

Theorem: Any g in $\tilde{\mathring{H}}_{\mathrm{cc}}$ admits the decomposition

$$
\begin{aligned}
g &= \mathrm{hess}(S^{(1)}_{\mathrm{cc}}\, g) \\
&\quad + \mathrm{def}(S^{(2)}_{\mathrm{cc}}\, g) \\
&\quad + D_{\mathrm{cc}}\, \mathrm{inc}\, g.
\end{aligned}
$$

Regular decompositions of the spaces



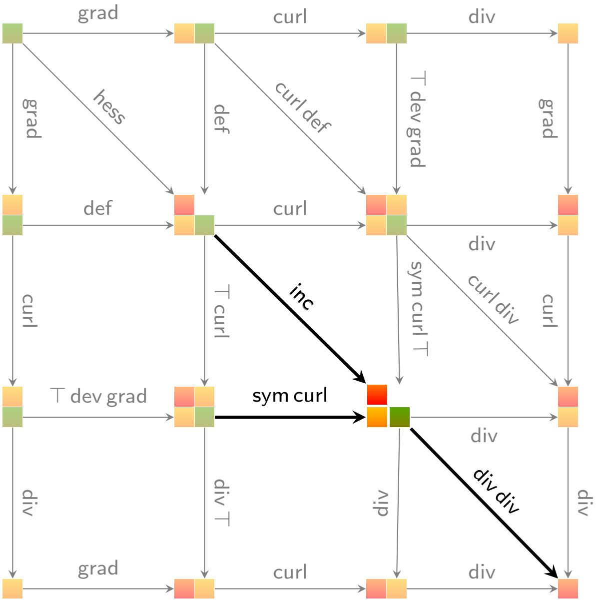
$\mathring{H}^1$	(the good)
L_2	(the bad)
$\tilde{\tilde{H}}^{-1}$	(the ugly)

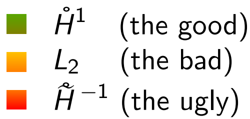
The $\tilde{\tilde{H}}_{\mathrm{cd}}$ case

Theorem: Any τ in $\tilde{\tilde{H}}_{\mathrm{cd}}$ admits the decomposition

$$\begin{aligned}\tau = {} & \operatorname{curl}\operatorname{def}(S^{(1)}_{\mathrm{cd}}\,\tau) \\ & + \operatorname{curl}(S^{(2)}_{\mathrm{cd}}\,\tau) \\ & + \top \operatorname{dev}\operatorname{grad}(S^{(3)}_{\mathrm{cd}}\,\tau) \\ & + D_{\mathrm{cd}} \operatorname{curl}\operatorname{div}\tau.\end{aligned}$$

Regular decompositions of the spaces





The $\mathring{\tilde{H}}_{\mathrm{dd}}$ case

Theorem: *Any* σ *in* $\mathring{\tilde{H}}_{\mathrm{dd}}$ *admits the decomposition*

$$\sigma = \mathrm{inc}(S_{\mathrm{dd}}^{(1)}\,\sigma) + \mathrm{sym\,curl}(S_{\mathrm{dd}}^{(2)}\,\sigma) + D_{\mathrm{dd}}\,\mathrm{div\,div}\,\sigma.$$

Outline

Diagram for the case **without boundary conditions**

$$
\begin{tikzcd}
H^1 \arrow[r, "\mathrm{grad}"] \arrow[d, "\mathrm{grad}"] & H(\mathrm{curl}) \arrow[r, "\mathrm{curl}"] \arrow[d, "\mathrm{def}"] & H(\mathrm{div}) \arrow[r, "\mathrm{div}"] \arrow[d, "\frac{1}{2}\top\,\mathrm{dev\,grad}"] & L_2 \arrow[d, "\frac{1}{3}\mathrm{grad}"] \\
H(\mathrm{curl}) \arrow[r, "\mathrm{def}"] \arrow[d, "\mathrm{curl}"] & H_{\mathrm{cc}} \arrow[r, "\mathrm{curl}"] \arrow[d, "\top\,\mathrm{curl}"] & H_{\mathrm{cd}} \arrow[r, "\mathrm{div}"] \arrow[d, "\mathrm{sym\,curl}\,\top"] & H^{-1}(\mathrm{curl}) \arrow[d, "\frac{1}{2}\mathrm{curl}"] \\
H(\mathrm{div}) \arrow[r, "\frac{1}{2}\mathrm{dev\,grad}"] \arrow[d, "\mathrm{div}"] & H_{\mathrm{cd}\top} \arrow[r, "\mathrm{sym\,curl}"] \arrow[d, "\mathrm{div}\,\top"] & H_{\mathrm{dd}} \arrow[r, "\mathrm{div}"] \arrow[d, "\mathrm{div}"] & H^{-1}(\mathrm{div}) \arrow[d, "\mathrm{div}"] \\
L_2 \arrow[r, "\frac{1}{3}\mathrm{grad}"] & H^{-1}(\mathrm{curl}) \arrow[r, "\frac{1}{2}\mathrm{curl}"] & H^{-1}(\mathrm{div}) \arrow[r, "\mathrm{div}"] & H^{-1}
\end{tikzcd}
$$

Prior results extend to this diagram also. (We can also start with $H^1/\mathcal{P}_1$ instead of H^1 and mod out appropriate subspaces as in the next slide.)

Duality

$$
\begin{tikzcd}
\mathring{H}^1 \arrow[rr] \arrow[dr, "*"] \arrow[dd] & & \mathring{H}(\mathrm{curl}) \arrow[rr] \arrow[dr, "*"] \arrow[dd] & & \mathring{H}(\mathrm{div}) \arrow[rr] \arrow[dr, "*"] \arrow[dd] & & L_{2,\mathbb{R}} \arrow[dr, "*"] \arrow[dd] & \\
& H^{-1} & & H^{-1}(\mathrm{div}) \arrow[ll] & & H^{-1}(\mathrm{curl}) \arrow[ll] & & L_2/\mathbb{R} \arrow[ll] \\
\mathring{H}(\mathrm{curl}) \arrow[rr] \arrow[dr, "*"] \arrow[dd] & & \mathring{\tilde{H}}_{\mathrm{cc}} \arrow[rr] \arrow[dr, "*"] \arrow[dd] & & \mathring{\tilde{H}}_{\mathrm{cd}} \arrow[rr] \arrow[dr, "*"] \arrow[dd] & & \mathring{\tilde{H}}^{-1}_{\mathcal{RT}}(\mathrm{curl}) \arrow[dr, "*"] \arrow[dd] & \\
& H^{-1}(\mathrm{div}) \arrow[uu] & & H_{\mathrm{dd}} \arrow[ll] \arrow[uu] & & H_{\mathrm{cd}\top} \arrow[ll] \arrow[uu] & & H(\mathrm{div})/\mathcal{RT} \arrow[ll] \arrow[uu] \\
\mathring{H}(\mathrm{div}) \arrow[rr] \arrow[dr, "*"] \arrow[dd] & & \mathring{\tilde{H}}_{\mathrm{cd}\top} \arrow[rr] \arrow[dr, "*"] \arrow[dd] & & \mathring{\tilde{H}}_{\mathrm{dd}} \arrow[rr] \arrow[dr, "*"] \arrow[dd] & & \mathring{\tilde{H}}^{-1}_{\mathcal{ND}}(\mathrm{div}) \arrow[dr, "*"] \arrow[dd] & \\
& H^{-1}(\mathrm{curl}) \arrow[uu] & & H_{\mathrm{cd}} \arrow[ll] \arrow[uu] & & H_{\mathrm{cc}} \arrow[ll] \arrow[uu] & & H(\mathrm{curl})/\mathcal{ND} \arrow[ll] \arrow[uu] \\
L_{2,\mathbb{R}} \arrow[rr] \arrow[dr, "*"] & & \mathring{\tilde{H}}^{-1}_{\mathcal{RT}}(\mathrm{curl}) \arrow[rr] \arrow[dr, "*"] & & \mathring{\tilde{H}}^{-1}_{\mathcal{ND}}(\mathrm{div}) \arrow[rr] \arrow[dr, "*"] & & \mathring{\tilde{H}}^{-1}_{\mathcal{P}_1} \arrow[dr, "*"] & \\
& L_2/\mathbb{R} \arrow[uu] & & H(\mathrm{div})/\mathcal{RT} \arrow[ll] \arrow[uu] & & H(\mathrm{curl})/\mathcal{ND} \arrow[ll] \arrow[uu] & & H^1/\mathcal{P}_1 \arrow[ll] \arrow[uu]
\end{tikzcd}
$$

Duality identities

1. $(\mathring{H}^1)^* = H^{-1}$
2. $(H^1)^* = \mathring{\tilde{H}}^{-1}$
3. $\mathring{H}(\text{div})^* = H^{-1}(\text{curl})$
4. $\mathring{H}(\text{curl})^* = H^{-1}(\text{div})$
5. $H(\text{div})^* = \mathring{\tilde{H}}^{-1}(\text{curl})$
6. $H(\text{curl})^* = \mathring{\tilde{H}}^{-1}(\text{div})$
7. $\mathring{\tilde{H}}^*_{\mathrm{cc}} = H_{\mathrm{dd}}$
8. $\mathring{\tilde{H}}^*_{\mathrm{cd}} = H_{\mathrm{cd}\top}$
9. $\mathring{\tilde{H}}^*_{\mathrm{dd}} = H_{\mathrm{cc}}$

Conclusion

- Operator $\operatorname{curl}\operatorname{div}$ is natural for viscous stresses of Stokes flow with $H(\operatorname{div})$ velocity.
- Simple *nt*-continuous matrix finite elements can be designed: [G+Lederer+Schöberl 2019].
- The MCS method yield optimal rates for fluid stresses and is Reynolds robust.
- MCS convergence analysis: [G+Lederer+Schöberl 2020], [G+Kogler+Lederer+Schöberl 2023].

- Other *nn*- and *tt*-continuous elements motivate study of accompanying Sobolev spaces.
- They feature 2nd order operators $\operatorname{curl}\operatorname{div}$, $\operatorname{div}\operatorname{div}$, and inc on matrix fields.

- Two commuting diagrams that give 2-complexes were seen. One is dual to the other.
- They contain new Sobolev spaces H_{cc}, H_{cd}, H_{dd} of matrix fields with low-regularity.
- Through regular decompositions, we understand these spaces better.
- Recent ArXiV preprint: [G+Hu+Schöberl 2025].