BROWN

Proximal Galerkin: A Unified Framework for Variational Problems with Inequality Constraints

FEM@LLNL, May 5, 2026

Presented by

Brendan Keith

A Unified Framework for Variational Problems with Inequality Constraints

Agenda

Motivation

- Equality Constraints
- Inequality Constraints

Technical Approach

- Legendre Functions
- Bregman Proximal Point
- Proximal Galerkin

Error Analysis

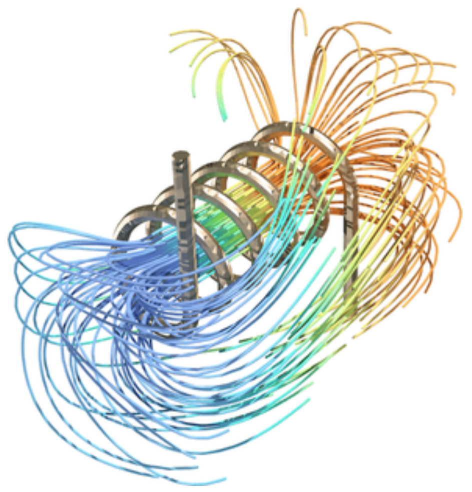
- Convergence Rates
- Mesh-Independence

Applications

- Obstacle Problem
- Contact Mechanics
- Variational Fracture
- Multi-Phase Flows
- Plasticity
- Eikonal Equation
- Monge–Ampère Equation
- Topology Optimization

...and more!

Equality **Constraints**

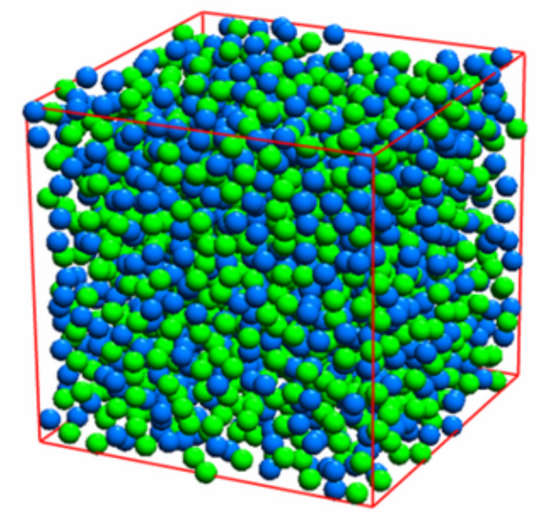


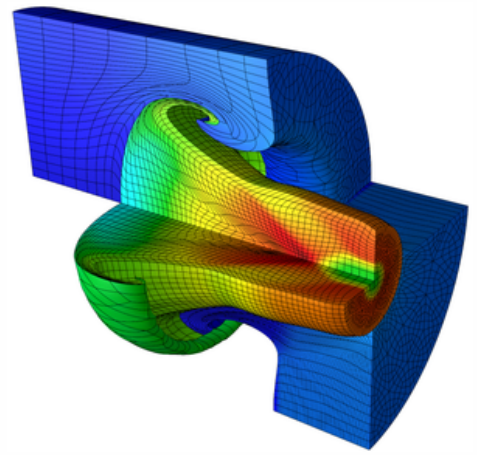
Electromagnetics

$$\nabla \cdot B = 0$$

Molecular Dynamics

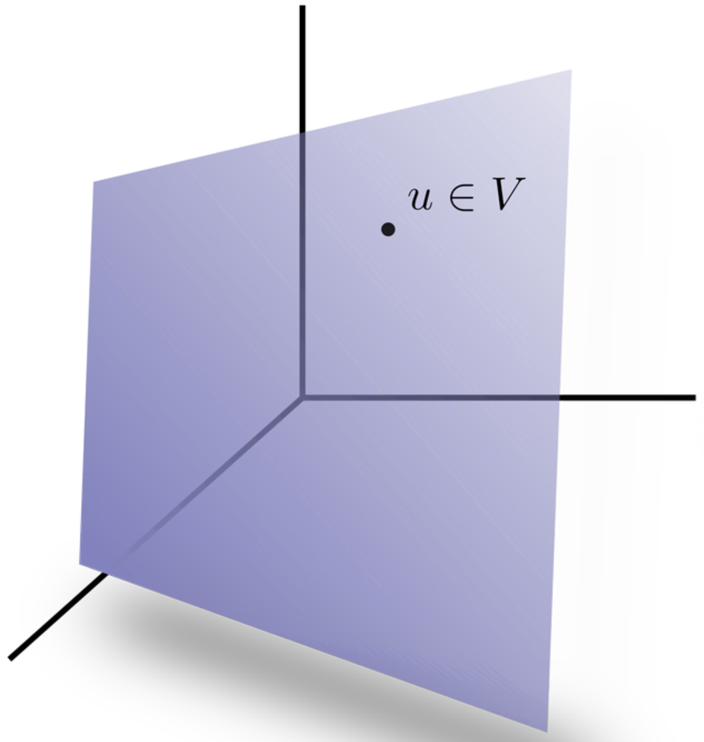
$$\frac{\mathrm{d}}{\mathrm{d}t}\left(E_{\mathrm{kin}} + E_{\mathrm{pot}} \right) = 0$$



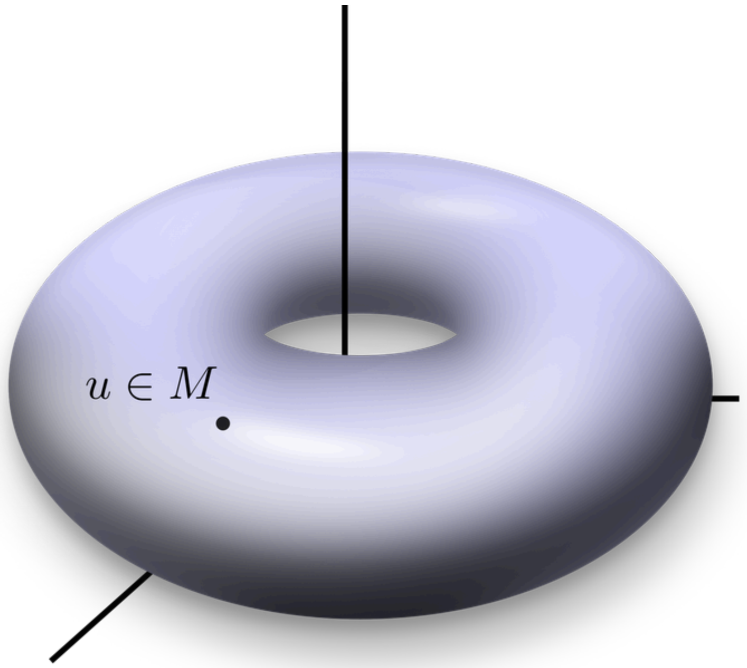


Multi-Material Transport

$$\partial_t u + \nabla \cdot F(u) = 0$$



Affine **Subspaces**



Closed Submanifolds

Shared Mathematical Foundation: Navigate and discretize subspaces and submanifolds

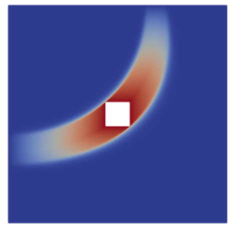
1st Class Numerical Methods

- *Mimetic finite differences*
- *Variational integrators*
- *Discrete exterior calculus*
- *Spectral methods*
- *Exponential integrators*
- *etc.*

Enable systematic, robust solutions

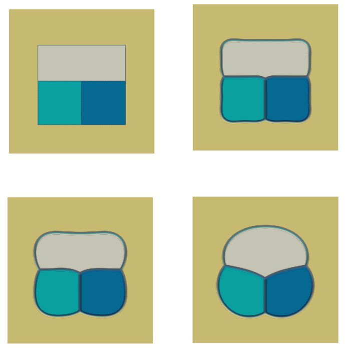
Inequality Constraints

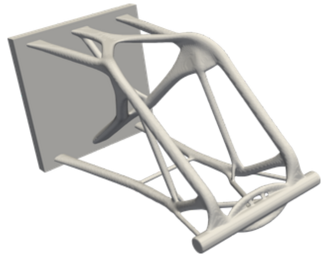
$$\rho \geq 0$$



Multi-Phase Flows

Maximum Principles



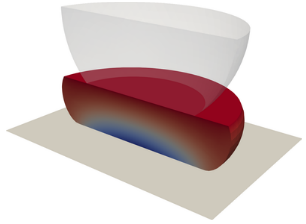


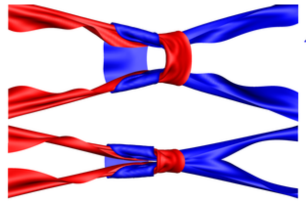
Optimal Design

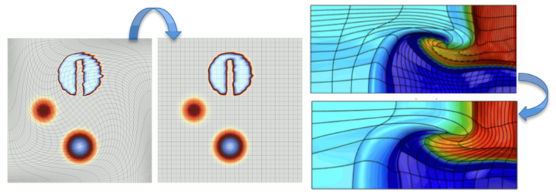
Densities need to stay non-negative

$$\det\left(\mathrm{D}\Phi\right) \geq 0$$

(Self-) Contact



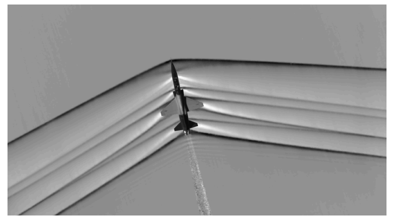




Mesh Tangling

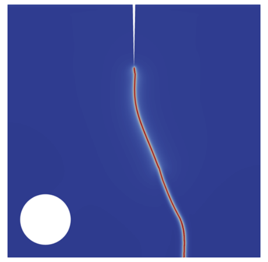
Matter cannot (self/inter-)penetrate

$$\frac{d}{dt} S \geq 0$$

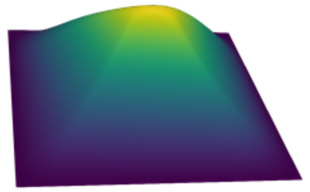


Shock Waves

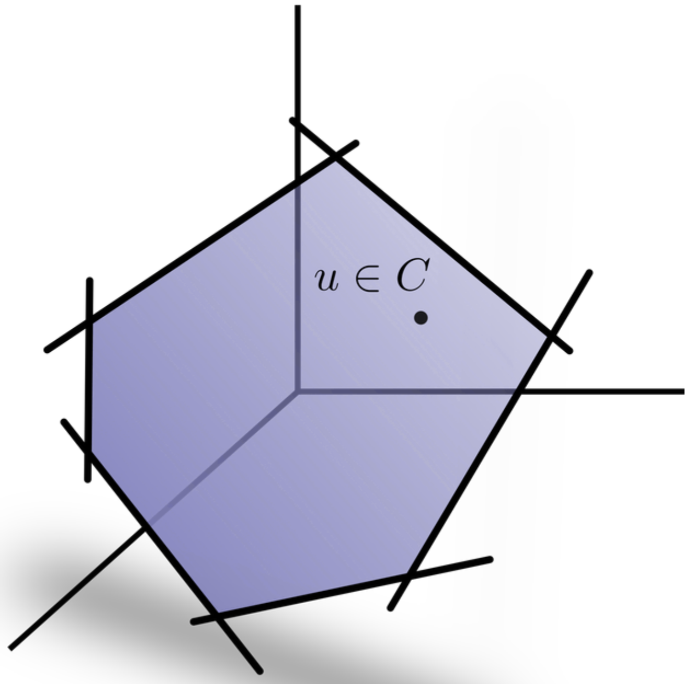
Fracture



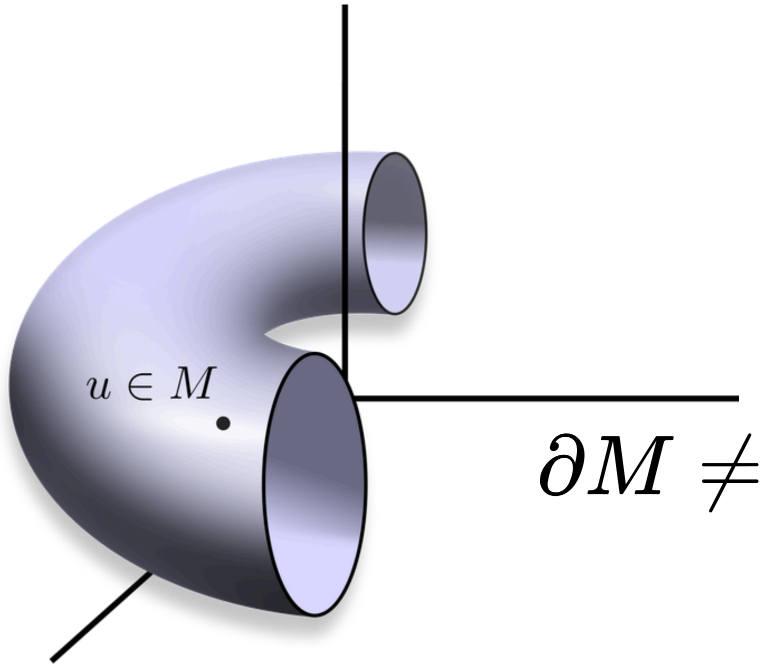
Elasto-Plasticity



Entropy can only increase



Convex **Polytopes**



$\partial M \neq \emptyset$

Manifolds with **Boundaries**

Required Mathematical Foundation: How to navigate and discretize convex sets and manifolds with *boundaries*?

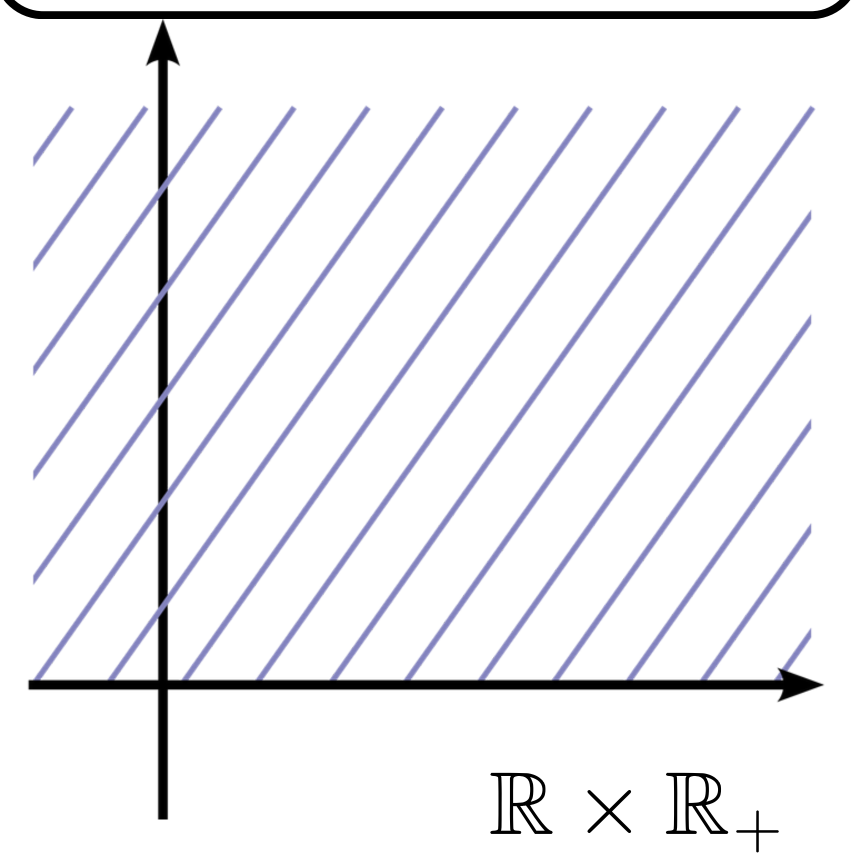
Solution Strategies

- *First-order methods (space/time)*
- *Artificial diffusion*
- *Flux-limiting*
- *Backtracking/remapping*
- *Nodal/quadrature point constraints*
- *Specialized basis functions*
- *Optimization-based limiters, etc.*

Practitioners face piecemeal of ad-hoc approaches

At world's end

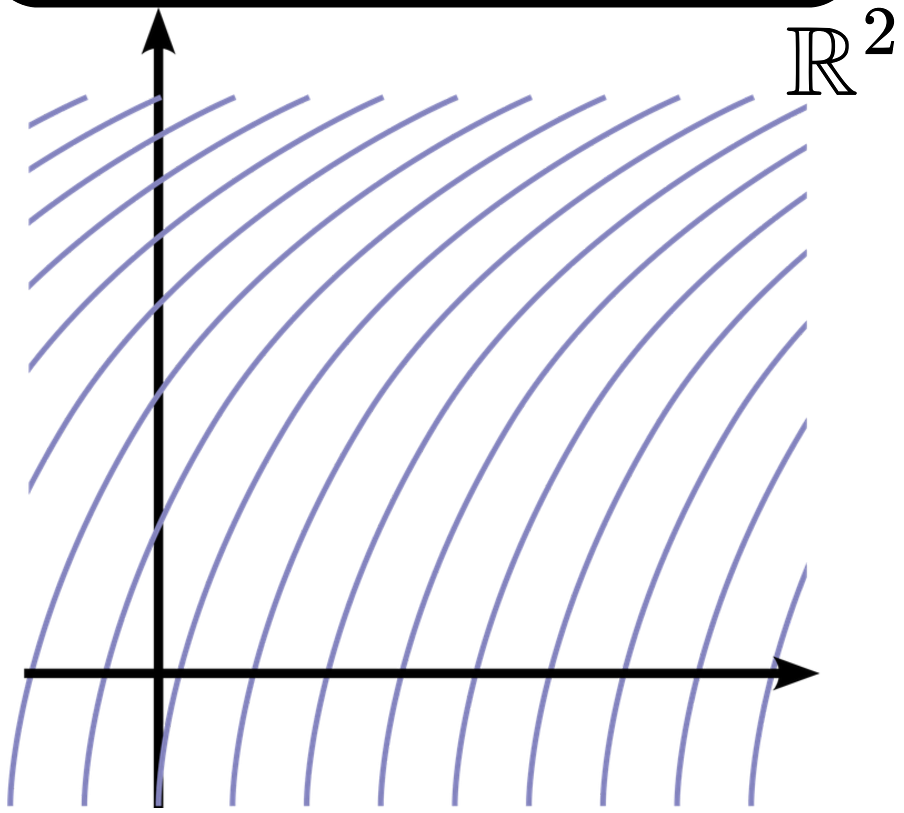
Idea: Transform problem to new coordinates!



$$\xrightarrow{(x,y)\mapsto(x,\ln y)}$$

How to generalize to more complicated sets?

Example: Logarithmic coordinates



Not so easy:
How to navigate now that we have to respect a boundary?

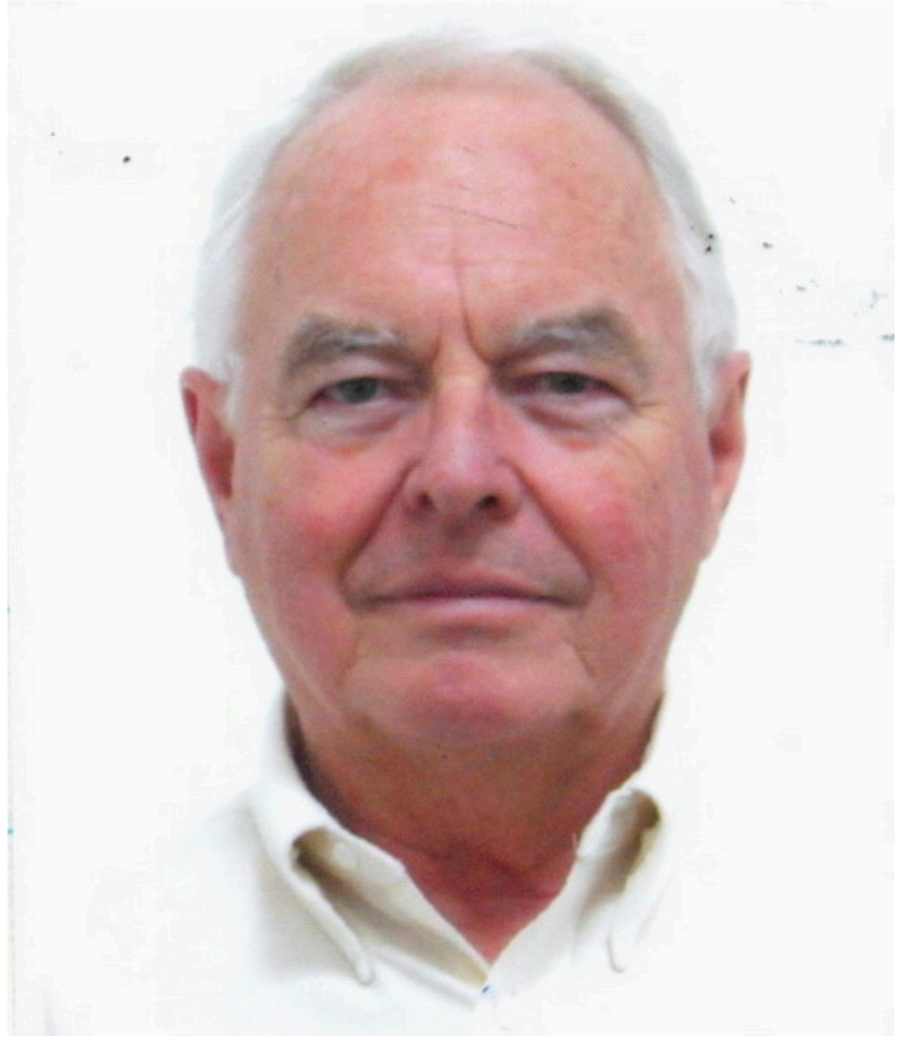
Legendre functions provide systematic machinery!

Legendre **Functions**

Definition

Let the essential domain of a function R be defined as $\operatorname{dom} R := \{a \in \mathbb{R}^m \mid R(a) < \infty\}$. We call a proper convex function a Legendre function $R : \mathbb{R}^m \to \mathbb{R} \cup \{+\infty\}$ if

- $\operatorname{int}(\operatorname{dom} R) \neq \varnothing$;
- R is differentiable on $\operatorname{int}(\operatorname{dom} R)$;
- $\lim_{t \to 0^+} \langle \nabla R(a + t(b - a)), b - a \rangle = -\infty$ for all $a \in \partial(\operatorname{dom} R)$ and $b \in \operatorname{int}(\operatorname{dom} R)$;
- R is strictly convex on $\operatorname{int}(\operatorname{dom} R)$.



R. Tyrrell Rockafellar

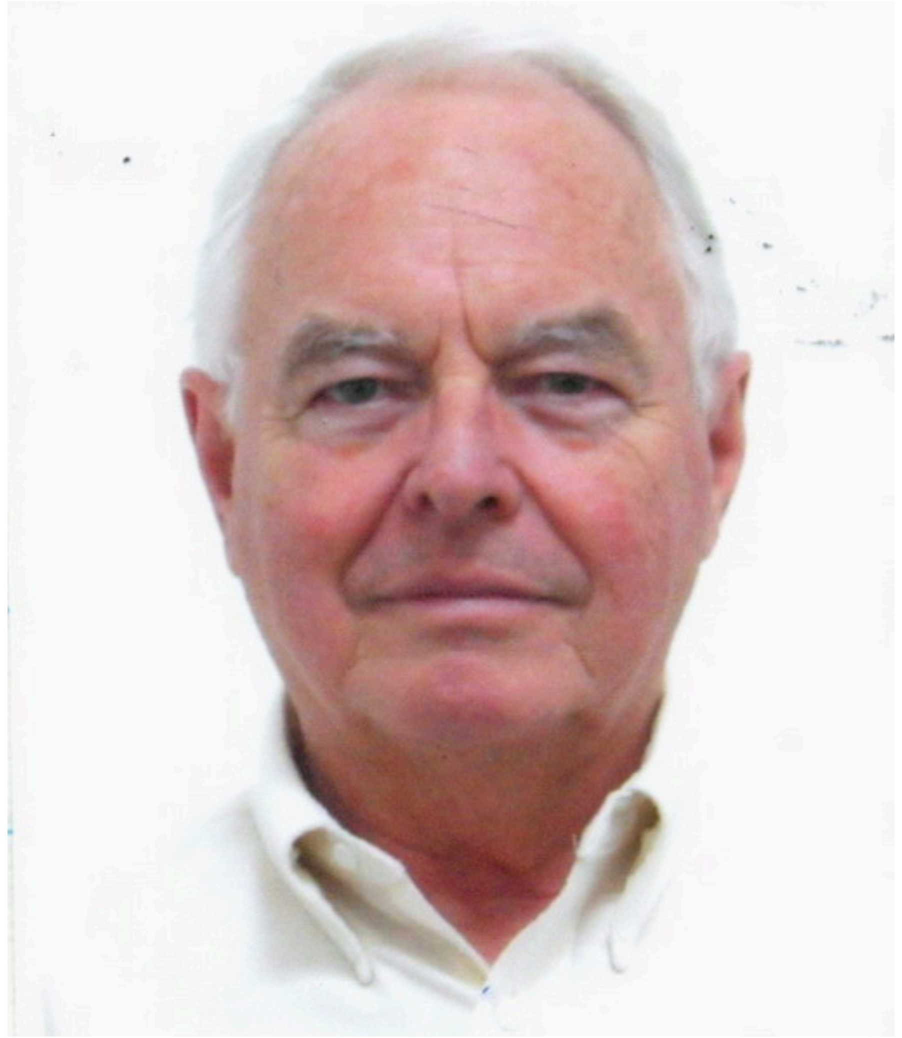
Special convex functions whose gradients are singular on the boundary of $\operatorname{dom} R$.

The geometry of a closed convex set $C \subset \mathbb{R}^m$ with non-empty interior can be encoded into a **Legendre function** with $\operatorname{dom} R = C$.

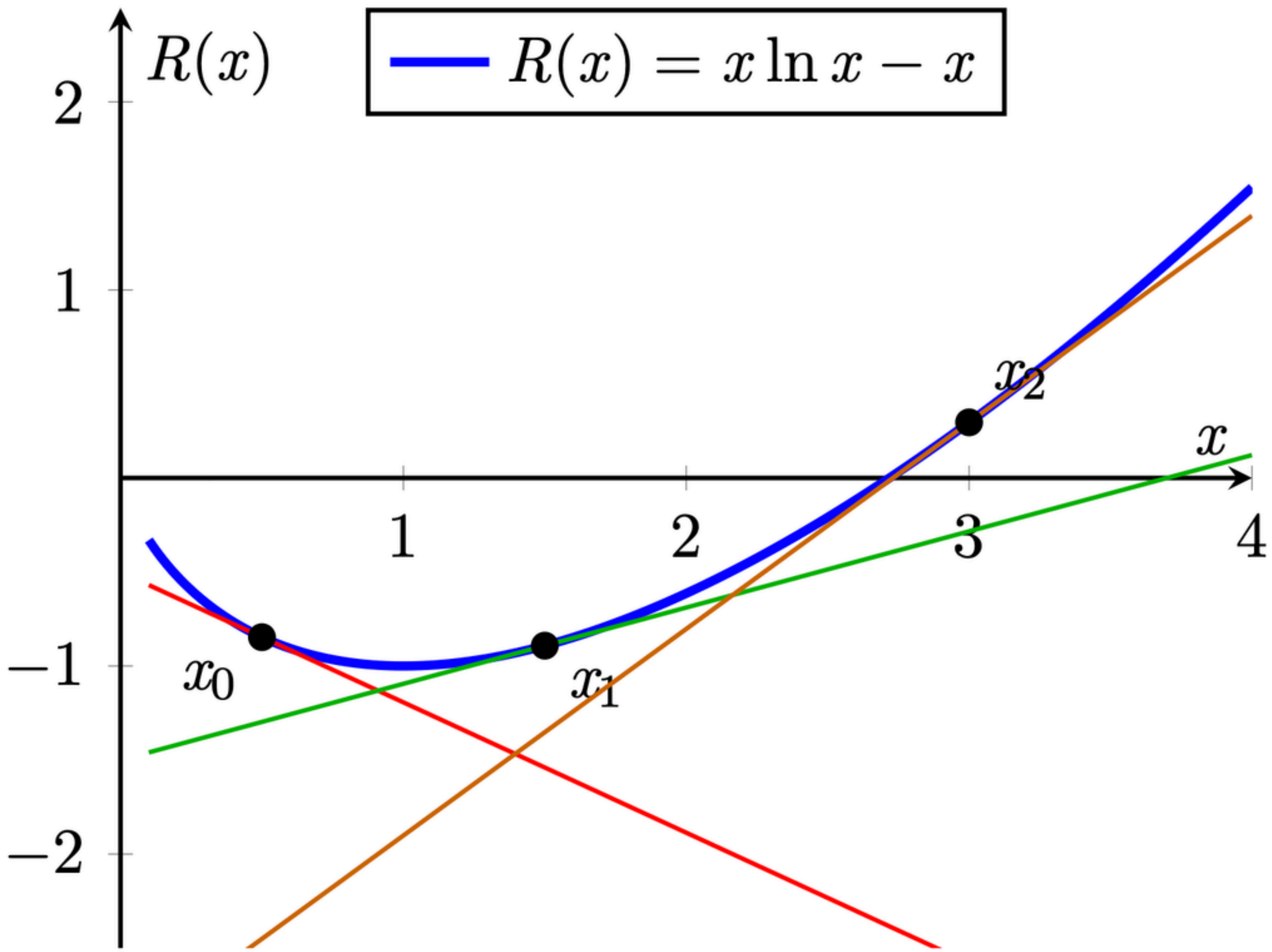
Theorem (Rockafellar, 1967)

A proper convex function R is a Legendre function if and only if its convex conjugate R^* is also a Legendre function. Moreover, $\nabla R\colon \operatorname{int}(\operatorname{dom} R) \to \operatorname{int}(\operatorname{dom} R^*)$ is a topological isomorphism with $(\nabla R)^{-1} = \nabla R^*$.

$$R^*(x^*) := \sup_{x \in \mathbb{R}^m} \{\langle x^*, x\rangle - R(x)\}$$



R. Tyrrell Rockafellar



R^* encodes all the information of the convex hull of $R(x)$'s epigraph in terms of its supporting hyperplanes

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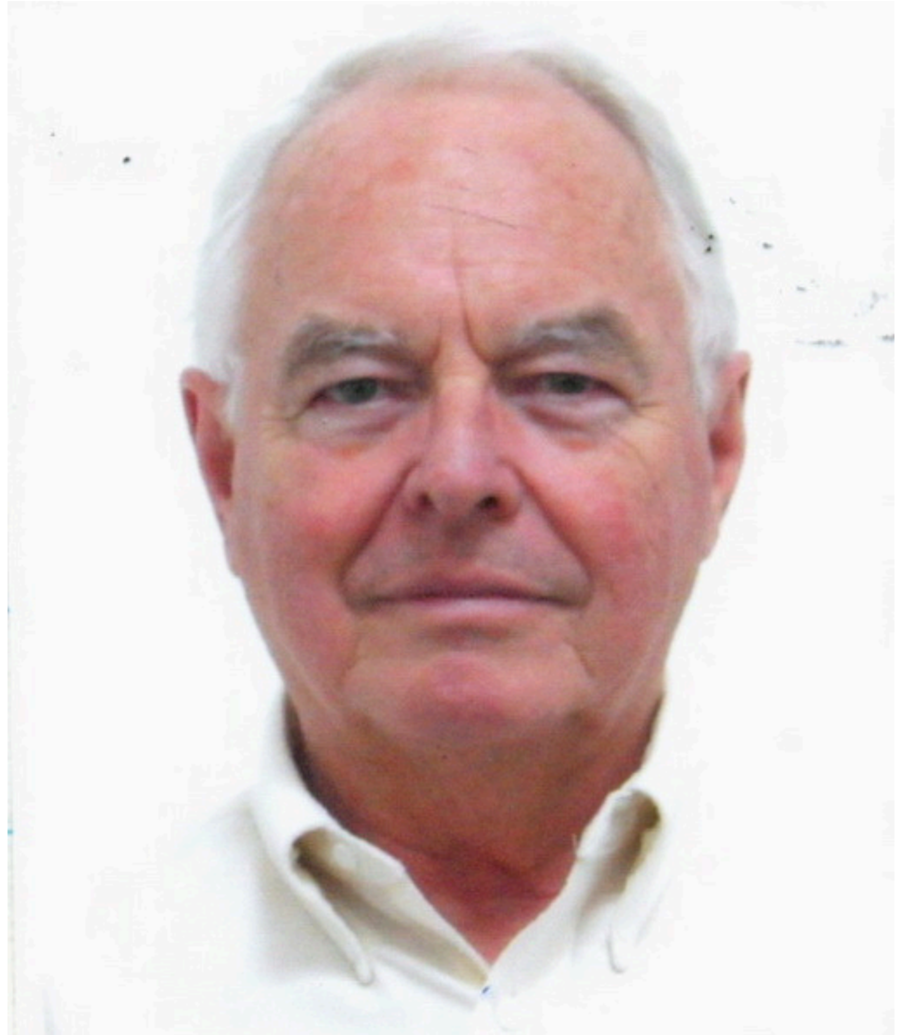
Fact: $\operatorname{dom} R^* = \mathbb{R}^m$ if and only if $R(a)/|a| \to +\infty$ as $|a| \to \infty$

$R(a)/|a| \to \infty$ as $|a| \to \infty$ and $\operatorname{dom} R = C$

$$\implies (\nabla R)^{-1} = \nabla R^* : \mathbb{R}^m \to \operatorname{int} C$$



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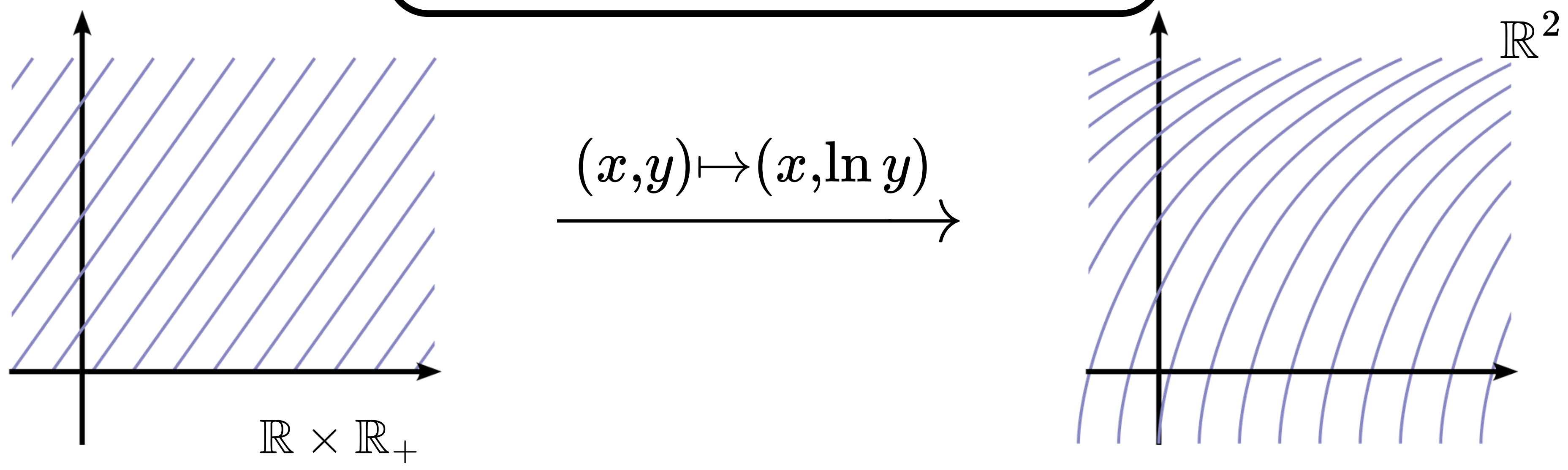
Geometry Preserving **Transformations**

Example: Logarithmic coordinates



Deriving the Coordinate Transformation

$$\begin{pmatrix} x \\ \ln y \end{pmatrix} = \nabla \underbrace{\left(\frac{x^2}{2} + y \ln y - y \right)}_{\text{Legendre function } R(x,y)}$$

The transformation is *singular*

$$\nabla R(x, y) = (x, \ln y)$$

But the inverse transformation is *smooth*!

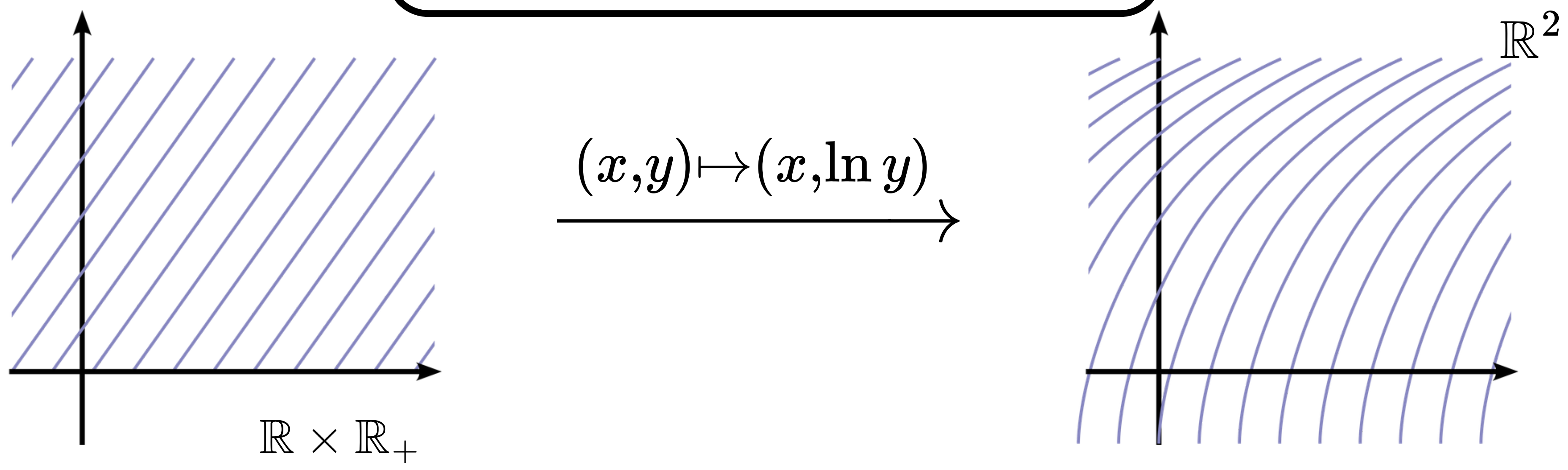
$$(\nabla R)^{-1}(x, y) = (x, \exp y)$$

Convex duality: the inverse is the gradient of the conjugate

$$(\nabla R)^{-1} = \nabla R^*$$

Geometry Preserving **Transformations**

Example: Logarithmic coordinates



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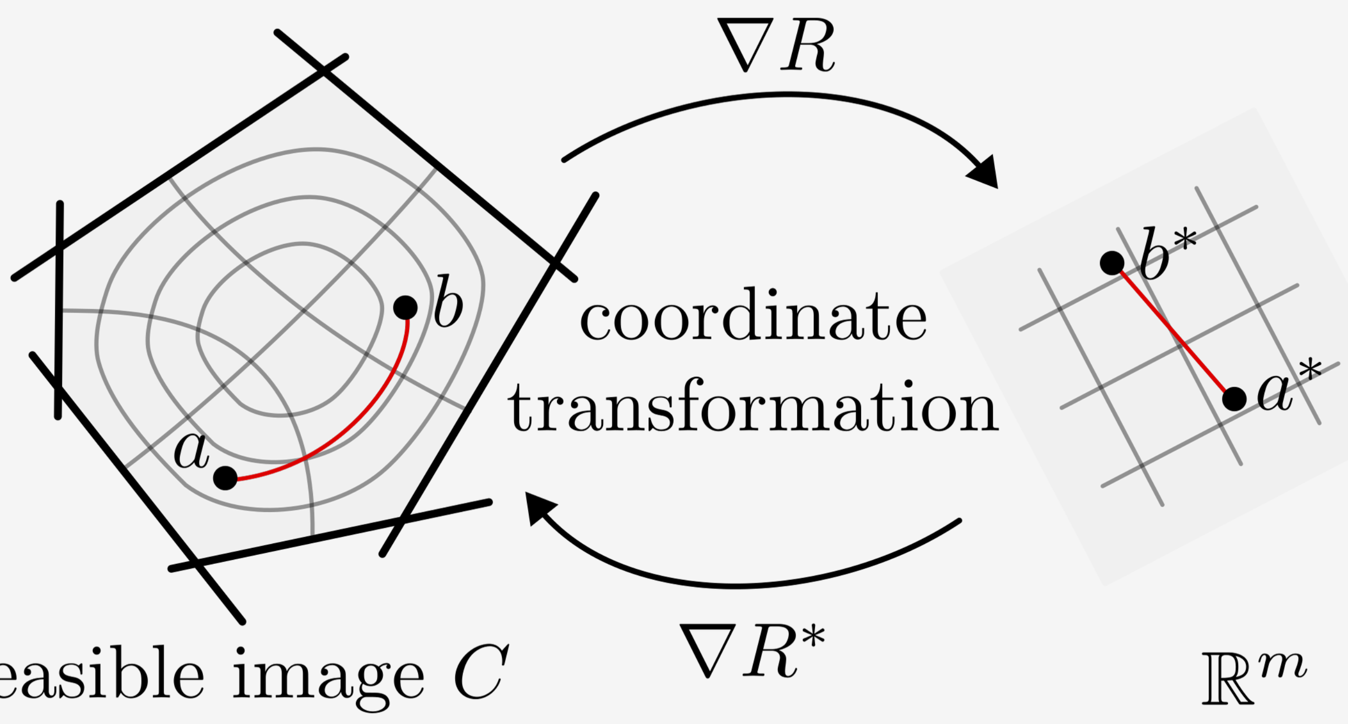
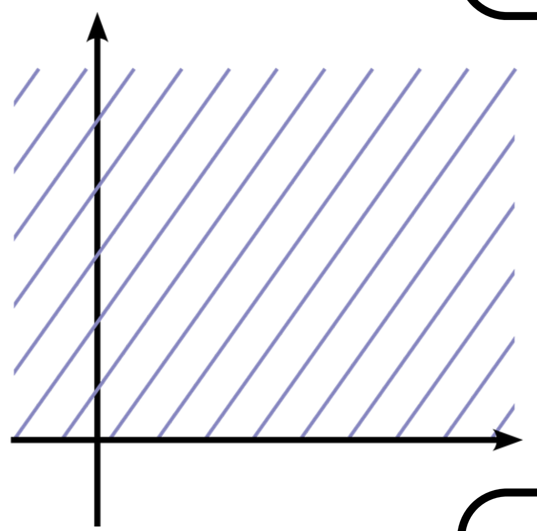
Legendre functions enable treating *general* **convex sets**

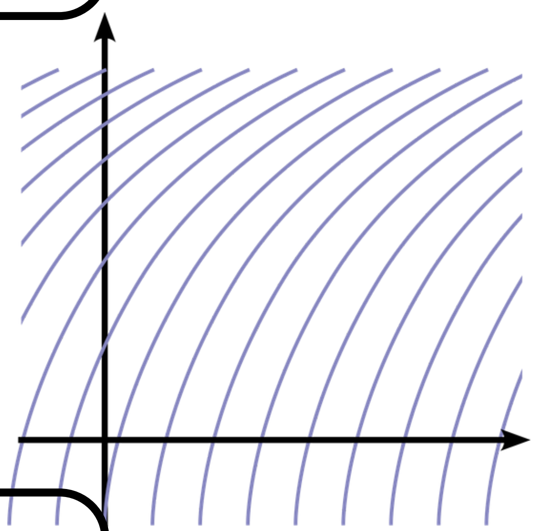


Geometry Preserving **Transformations**

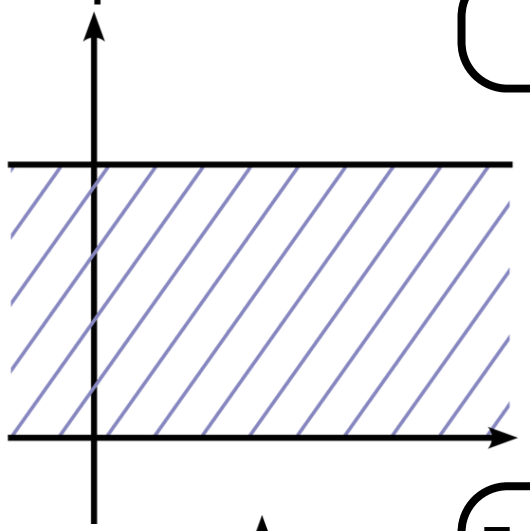
Example: Logarithmic coordinates



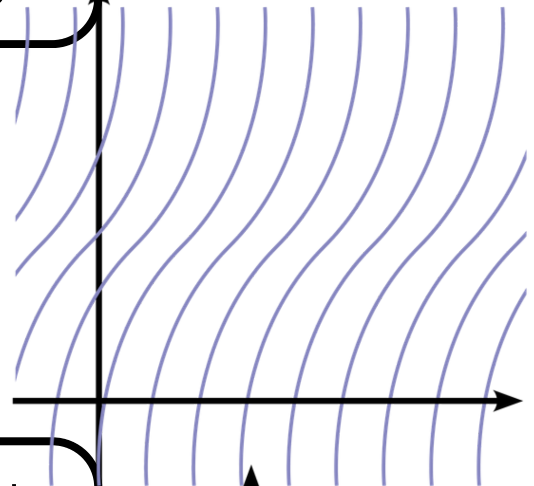
$$\xrightarrow{(x,y)\mapsto(x,\ln y)}$$



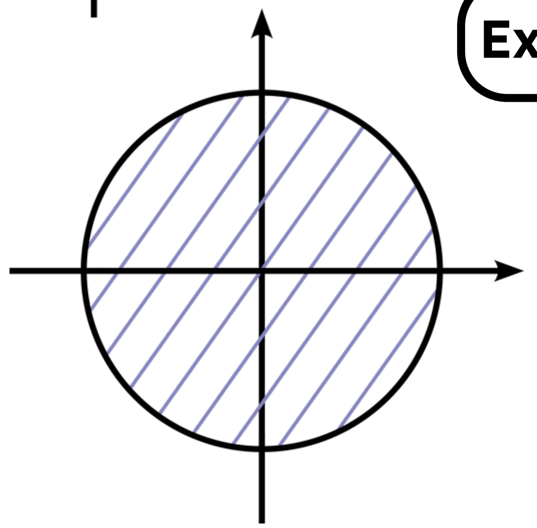
Example: Logistic coordinates



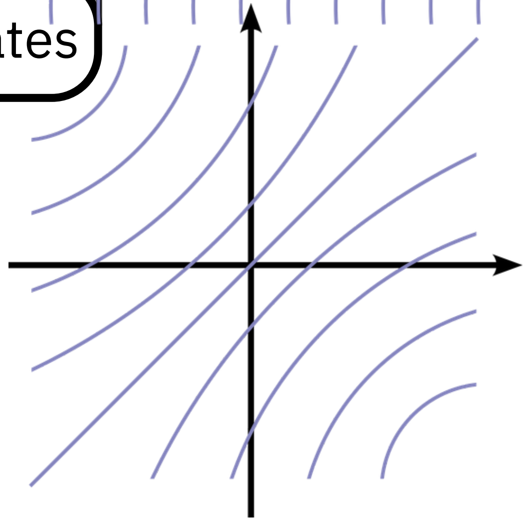
$$\xrightarrow{(x,y)\mapsto\left(x,\ln\frac{y}{1-y}\right)}$$



Example: Inverse radial coordinates



$$\xrightarrow{(x,y)\mapsto\frac{(x,y)}{\sqrt{1-r^2}}}$$



Deriving the Coordinate Transformation

$$\begin{pmatrix} x \\ \ln y \end{pmatrix} = \nabla \underbrace{\left(\frac{x^2}{2} + y\ln y - y\right)}_{\text{Legendre function } R(x,y)}$$

Coordinate Transformation

$$\begin{pmatrix} x \\ \ln\frac{y}{1-y} \end{pmatrix} = \nabla \underbrace{\left(\frac{x^2}{2} + y\ln y + (1-y)\ln(1-y)\right)}_{R(x,y)}$$

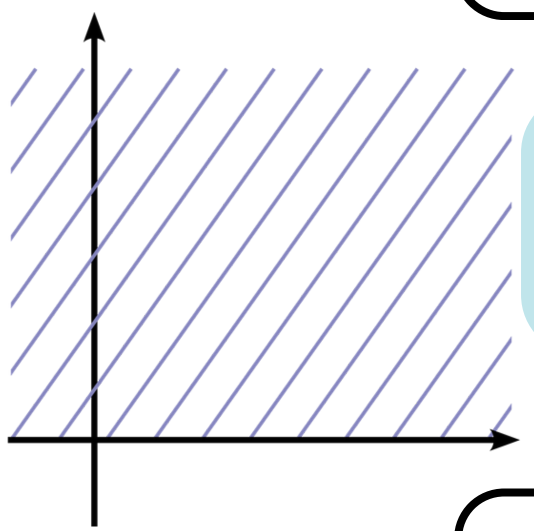
Coordinate Transformation

$$\frac{(x,y)^\top}{\sqrt{1-r^2}} = \nabla \underbrace{\left(-\sqrt{1-r^2}\right)}_{R(x,y)}$$

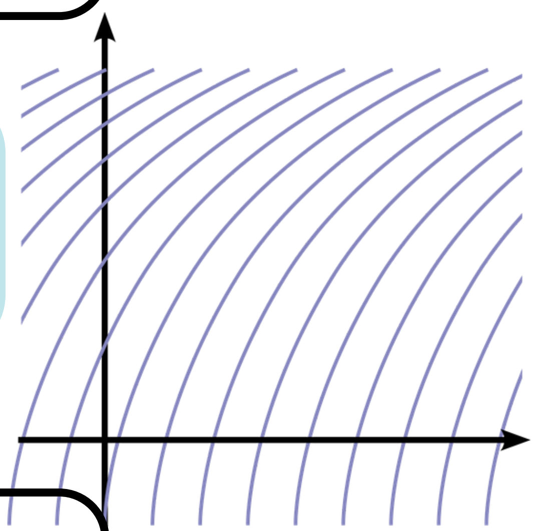
Geometry Preserving **Transformations**

Many more examples later!

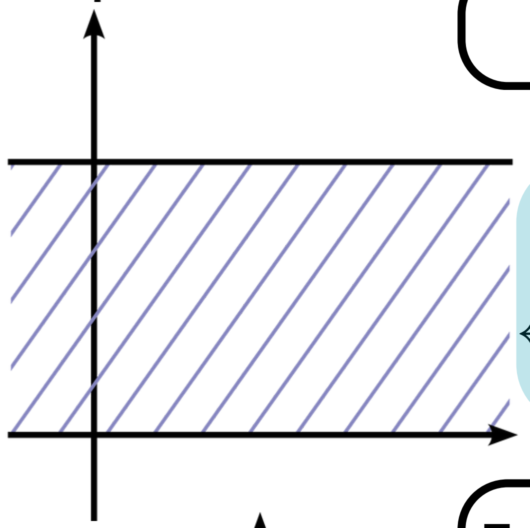
Example: Logarithmic coordinates



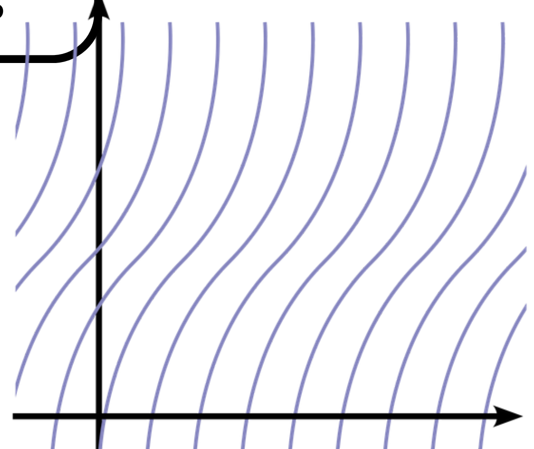
$$\nabla R^*(x,y)=(x,\exp y)$$



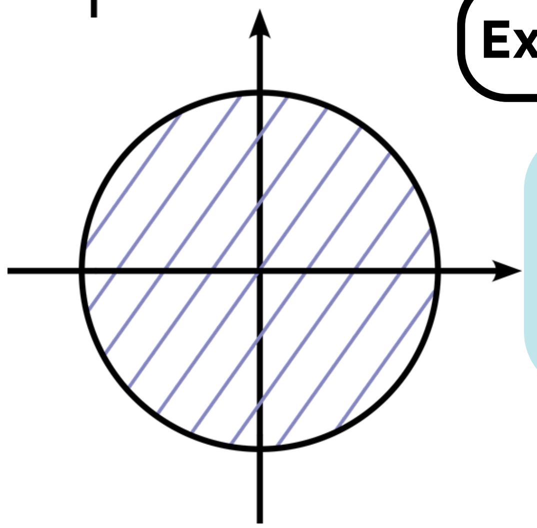
Example: Logistic coordinates



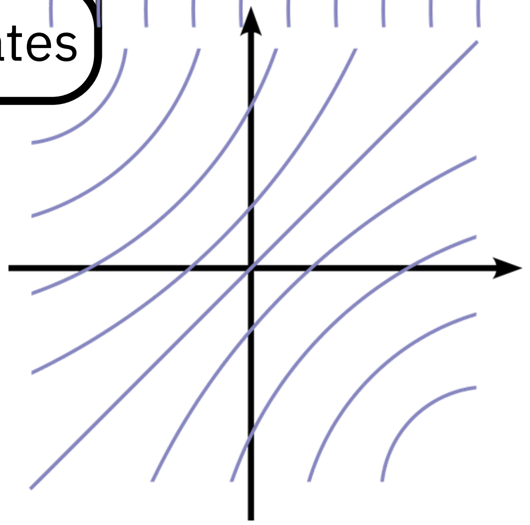
$$\nabla R^*(x,y)=\left(x,\frac{\exp y}{1+\exp y}\right)$$



Example: Inverse radial coordinates



$$\nabla R^*(x,y)=\frac{(x,y)}{\sqrt{1+r^2}}$$



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$$\begin{pmatrix} x \\ \ln y \end{pmatrix} = \nabla \underbrace{\left(\frac{x^2}{2}+y\ln y - y\right)}_{\text{Legendre function } R(x,y)}$$

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Coordinate Transformation

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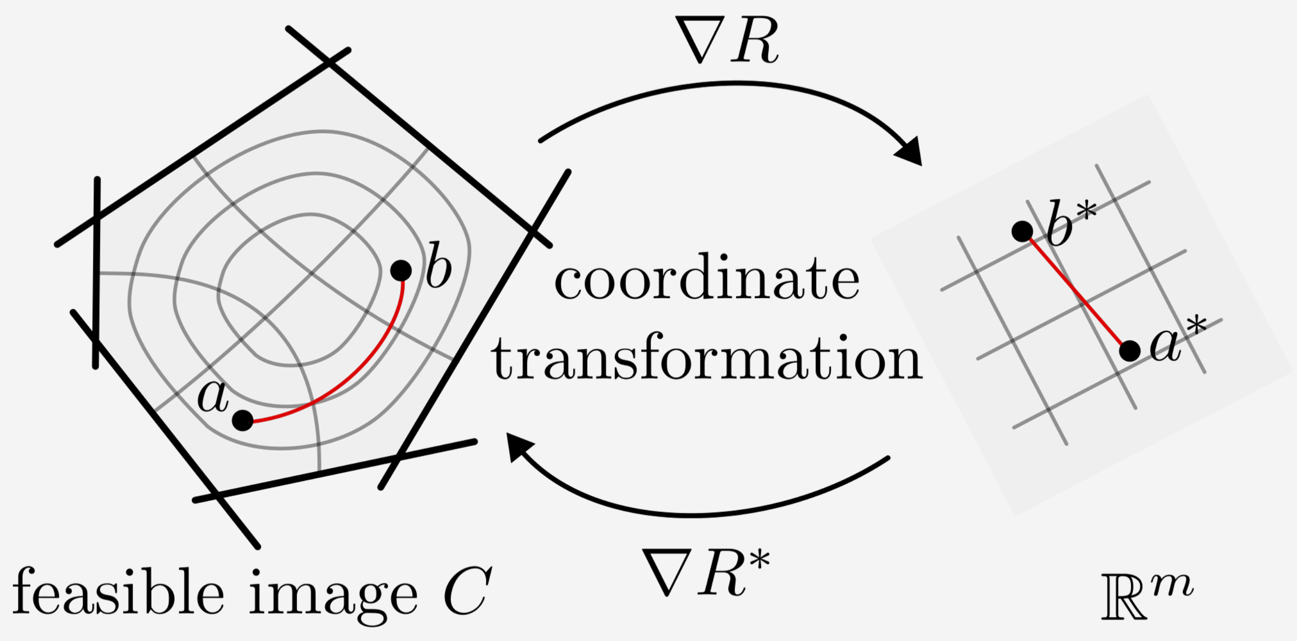


Metrics &
Gradient Flows

Legendre functions induce a Riemannian metric on $\operatorname{int} C$

THEORY

Manifolds

Treat $\operatorname{int} C$ as a Riemannian manifold with metric $\nabla^2 R$

Geodesics

Straight lines in $\mathbb{R}^m$ are mapped to **dual geodesics** γ in C

Gradient Flows

min F over C by following the **Riemannian gradient**

Gradient Flow: $\dfrac{\mathrm{d}u}{\mathrm{d}t} = -\nabla_R F(u), \quad \text{where } \nabla_R := (\nabla^2 R)^{-1}\nabla$

Riemannian Gradient

(Euclidean) Descent **Methods**

The (Euclidean) gradient flow of F is given by $\frac{\mathrm{d}u}{\mathrm{d}t} = -\nabla F(u), \quad u(0) = u_0$

Explicit Euler

$$\frac{u^{k+1} - u^k}{\alpha_k} = -\nabla F(u^k) \quad \Longleftrightarrow$$

Steepest Descent

$$u^{k+1} = u^k - \alpha_k \nabla F(u^k)$$

Linearized Subproblem

$$u^{k+1} = \min_u \; F(u^k) + \langle \nabla F(u^k), u - u_k \rangle + \frac{1}{2\alpha_k} \|u - u^k\|^2$$

(Euclidean) Descent **Methods**

The (Euclidean) gradient flow of F is given by $\frac{\mathrm{d}u}{\mathrm{d}t} = -\nabla F(u), \quad u(0) = u_0$

What if we don't linearize F in the subproblem?

Linearized Subproblem

$$u^{k+1} = \min_u \; F(u^k) + \langle \nabla F(u^k), u - u_k \rangle + \frac{1}{2\alpha_k} \|u - u^k\|^2$$

(Euclidean) Descent **Methods**

The (Euclidean) gradient flow of F is given by $\dfrac{\mathrm{d}u}{\mathrm{d}t} = -\nabla F(u), \quad u(0) = u_0$

What if we don't linearize F in the subproblem?

"Proximal" Subproblem

$$u^{k+1} = \min_u \; F(u) + \frac{1}{2\alpha_k}\|u - u^k\|^2$$

(Euclidean) Descent **Methods**

The (Euclidean) gradient flow of F is given by $\dfrac{\mathrm{d}u}{\mathrm{d}t} = -\nabla F(u), \quad u(0) = u_0$

Implicit Euler

Proximal Point

$$\frac{u^{k+1} - u^k}{\alpha_k} = -\nabla F(u^{k+1}) \iff u^{k+1} = u^k - \alpha_k \nabla F(u^{k+1})$$

"Proximal" Subproblem

$$u^{k+1} = \min_u \; F(u) + \frac{1}{2\alpha_k}\|u - u^k\|^2$$

Riemannian Descent **Methods**

The Riemannian gradient flow is given by $\frac{\mathrm{d}u}{\mathrm{d}t} = -(\nabla^2 R(u))^{-1}\nabla F(u)$

Chain Rule

$$\nabla^2 R(u)\frac{\mathrm{d}u}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\nabla R(u) \quad \Longrightarrow$$

Implicit Euler

$$\frac{\nabla R(u^{k+1}) - \nabla R(u^k)}{\alpha_k} = -\nabla F(u^{k+1})$$

Proximal Subproblem

$$u^{k+1} = \min_u \; F(u) + \frac{1}{\alpha_k}D_R(u, u^k)$$

Bregman Divergence

"Bregman" Proximal Point

$$\nabla R(u^{k+1}) = \nabla R(u^k) - \alpha_k \nabla F(u^{k+1})$$

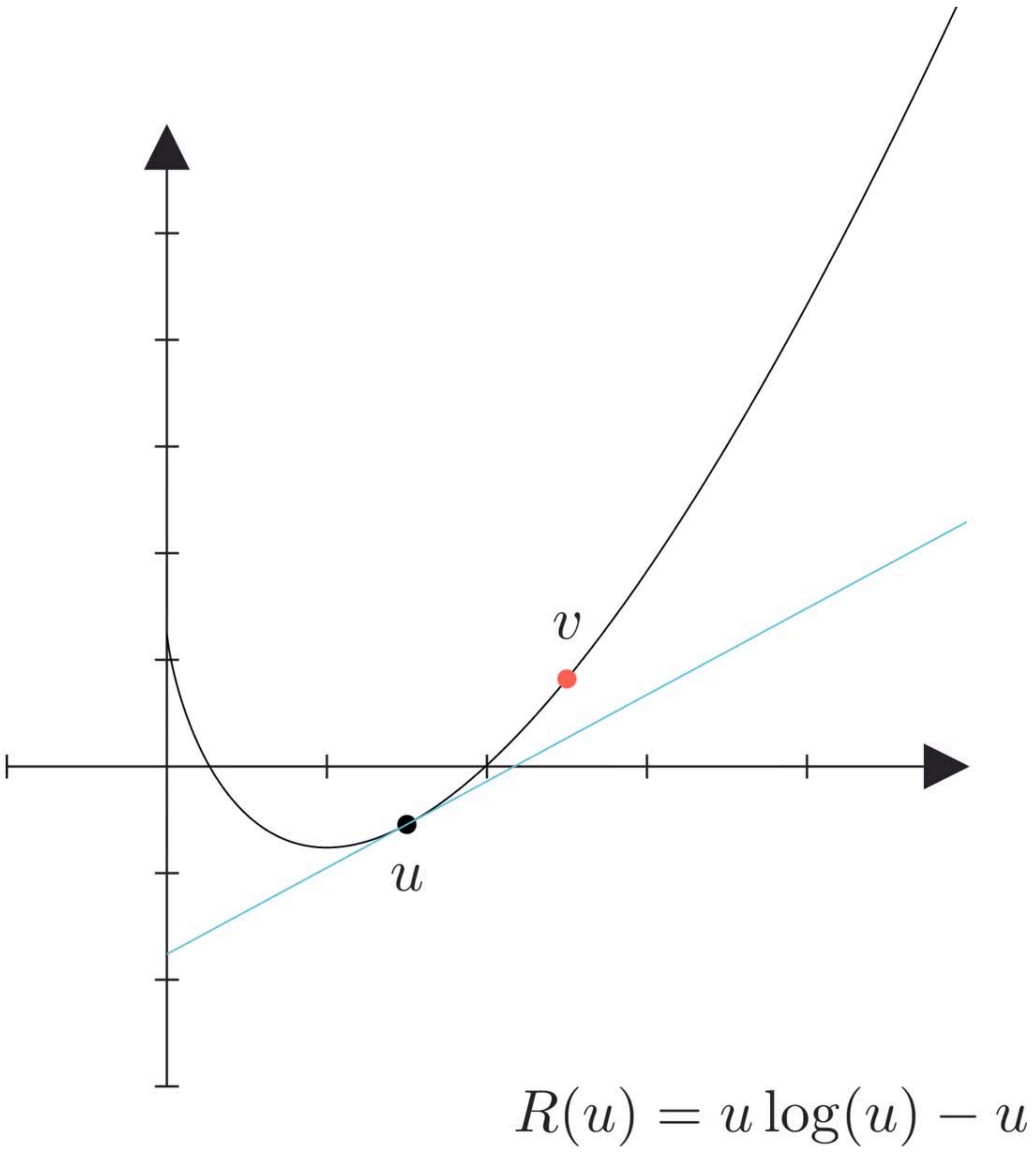
Bregman **Divergences**

Definition: Let $R\colon C \to \mathbb{R}$ be a *Legendre function*. Its **Bregman divergence** is

$$D_R(v,u) = R(v) - R(u) - \langle \nabla R(u), v - u \rangle$$

Example 1: The Bregman divergence of the function $R(u) = \frac{1}{2}\|u\|^2$ is (half) the Euclidean distance *squared*

$$D(v,u) = \frac{1}{2}\|v-u\|^2$$



Video courtesy of D. Kim

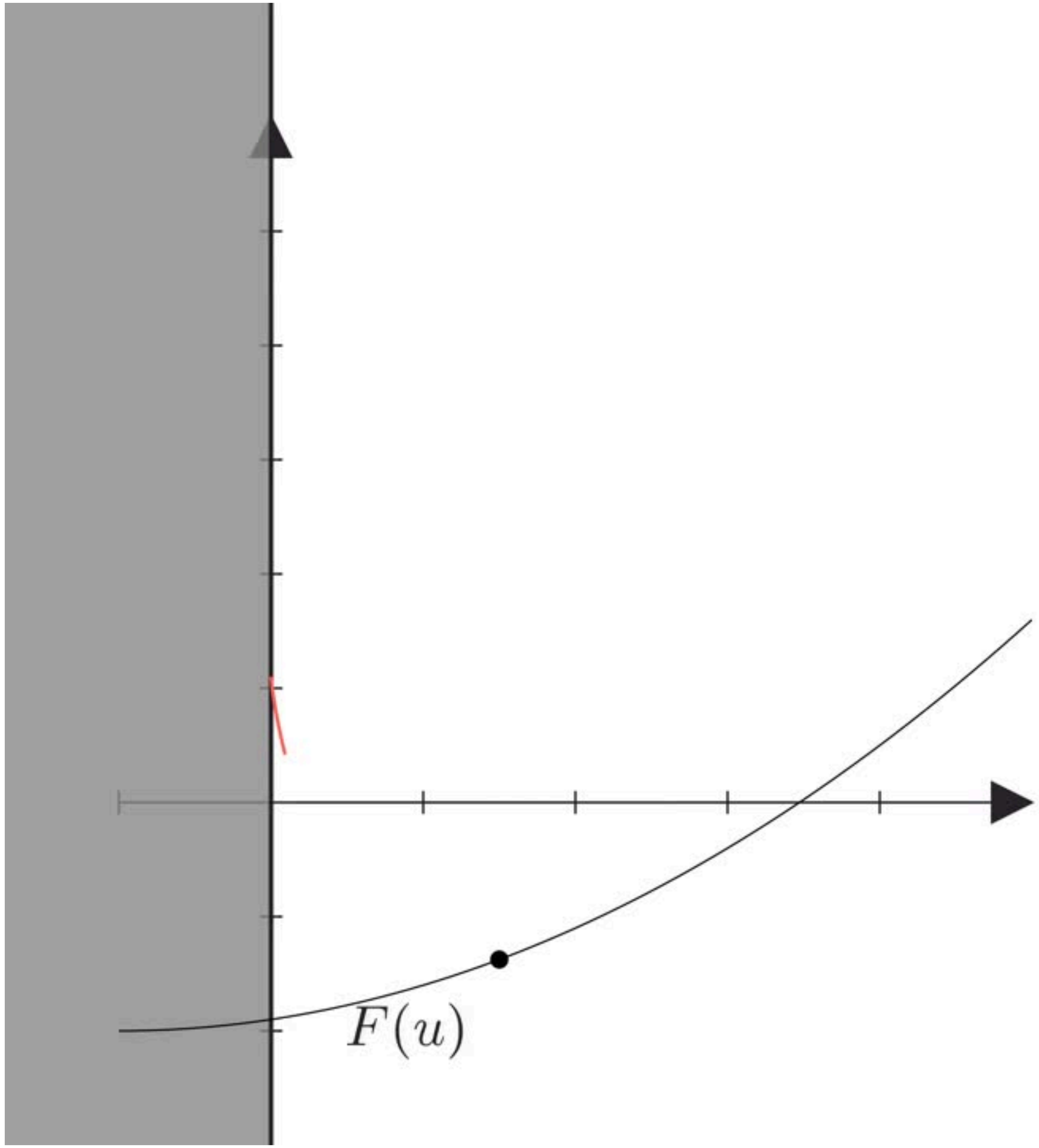
Takeaway **Message**

The **Bregman Proximal Point** algorithm,

$$u^{k+1} = \min_{u} \; F(u) + \alpha_k^{-1} D_R(u, u^k)$$

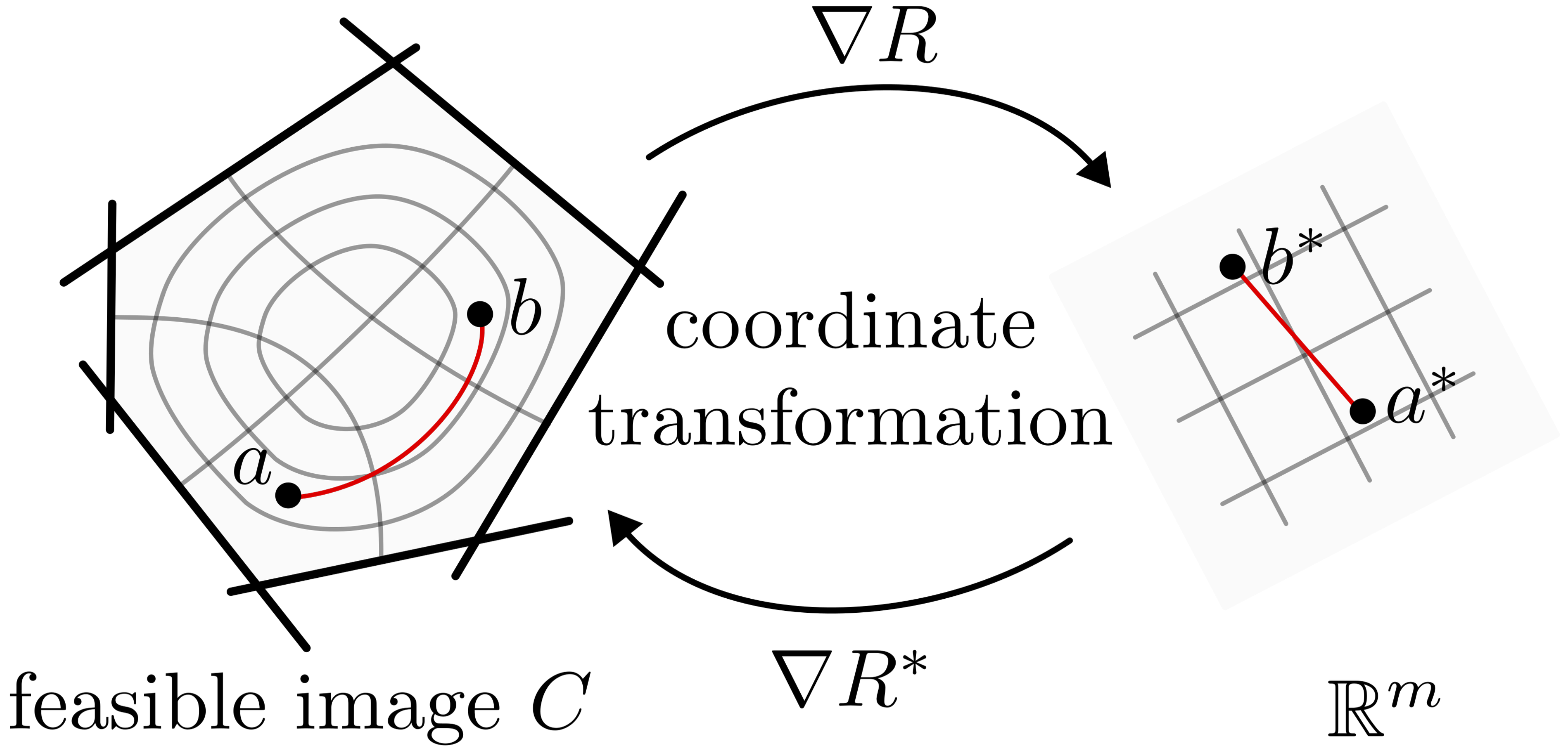
produces a sequence ***interior points*** $u \in \operatorname{int} C$ that converge to the minimizer

$$u^k \to \underset{u \in C}{\arg\min} \, F$$



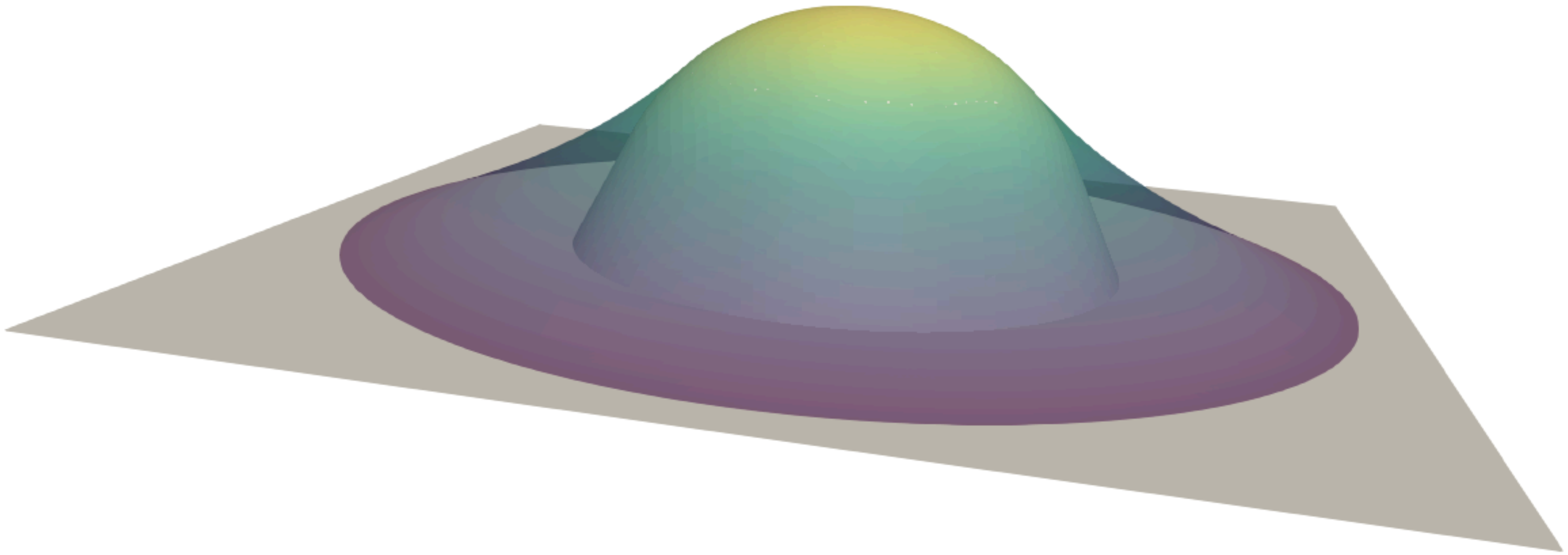
Video courtesy of D. Kim

How to use this for **Mechanics?**



Example

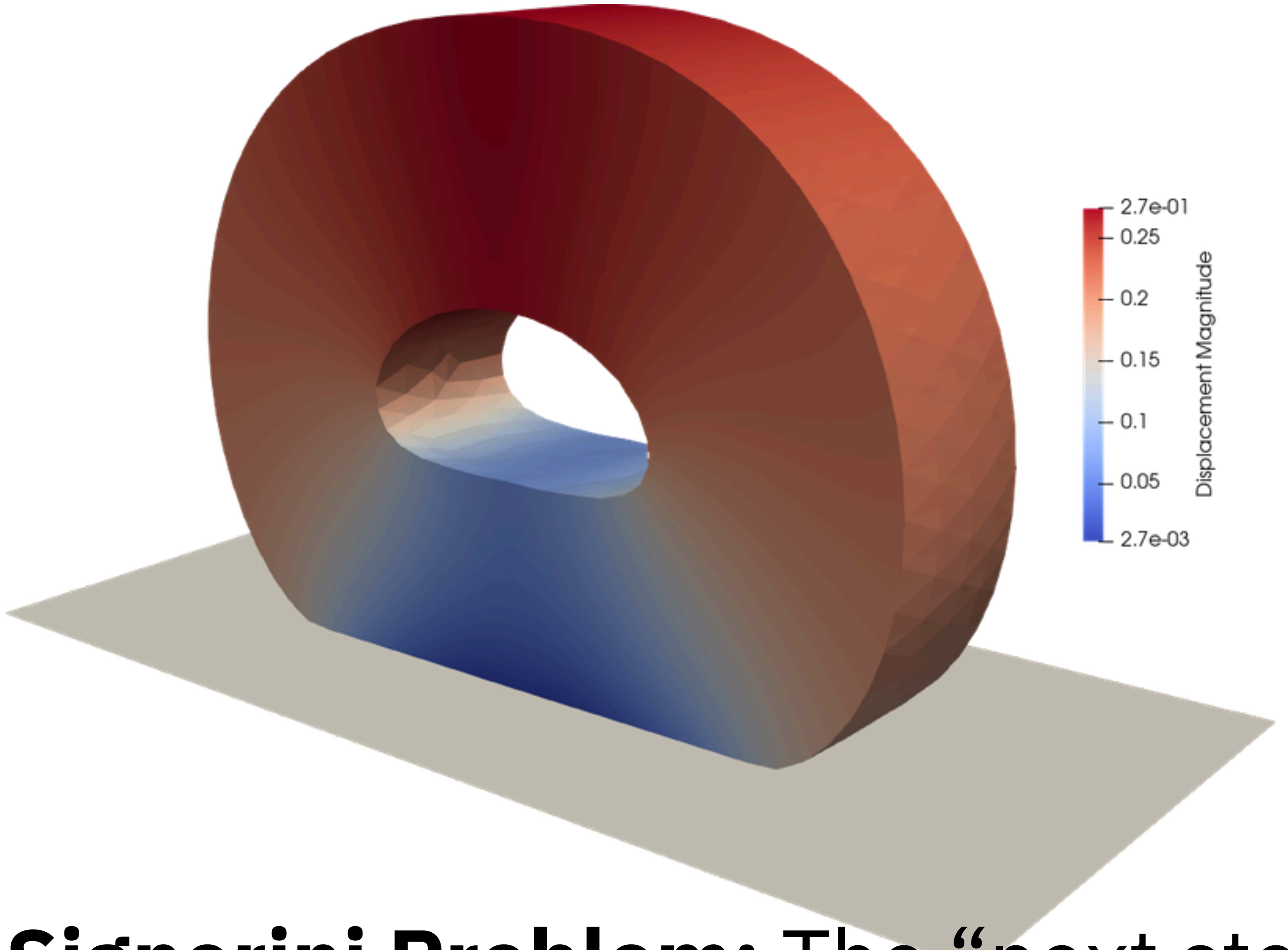
$$\min_{u \in K} J(u) = \frac{1}{2}\|\nabla u\|^2 \quad \text{where } K = \{u \in H_0^1(\Omega) \mid u \geq \phi \text{ a.e. in } \Omega\}$$



Obstacle Problem: The “hello world” of contact mechanics $K = \{u \in H_0^1 \mid u \geq \phi\}$

Example

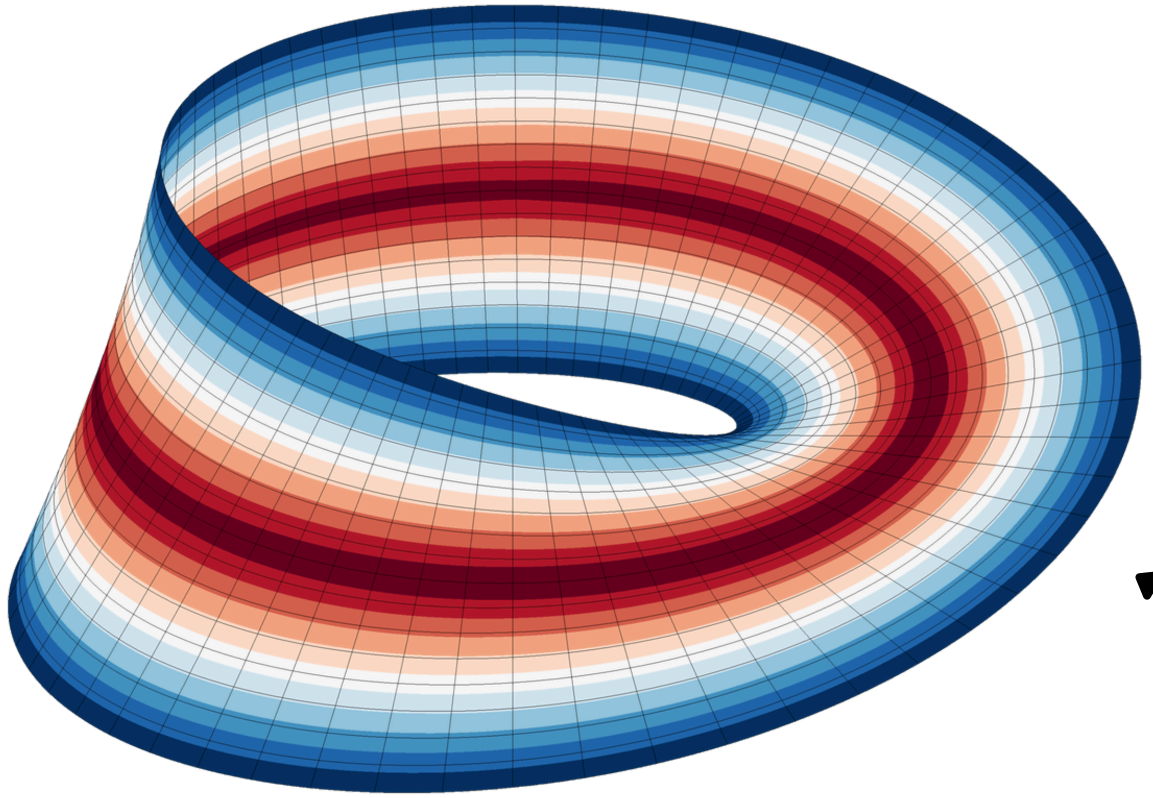
$$\min_{u \in K} J(u) \quad \text{where } K = \{u \in V \subset H^1(\Omega)^3 \mid u \cdot n \geq \phi \text{ a.e. on } \partial\Omega\}$$



Signorini Problem: The "next step" towards contact mechanics

Example

$$\min_{u \in K} - \int_{\Omega} u \, \mathrm{d}x \quad \text{where } K = \{u \in W_0^{1,\infty}(\Omega) \mid |\nabla u| \leq 1 \text{ a.e. in } \Omega\}$$



Möbius Strip

Vicosity solutions to the **eikonal equation:**

$$|\nabla u| = 1 \ \text{ in } \Omega, \qquad u = 0 \ \text{ on } \partial\Omega$$

Feasible Sets in **Mechanics**

$$K = \{v \in V \mid Bv(x) \in C(x) \text{ for almost every } x \text{ in a subset of } \overline{\Omega}\}.$$

	Feasible set K	Legendre function R	B	$\nabla R^*(\psi)$
Obstacle Problems	$\{u \geq \phi\}$	$(a-\phi)\ln(a-\phi)-(a-\phi)$	id	$\phi+\exp\psi$
Topology Optimization, **Variational Fracture**	$\{\phi_1 \leq u \leq \phi_2\}$	$(a-\phi_1)\ln(a-\phi_1)+(\phi_2-a)\ln(\phi_2-a)$	id	$\frac{\phi_1+\phi_2\exp\psi}{1+\exp\psi}$
	$\{\gamma u \geq \phi\}$	$(a-\phi)\ln(a-\phi)-(a-\phi)$	γ	$\phi+\exp\psi$
Contact	$\{(\gamma u)\cdot n \leq \phi\}$	$(\phi-a)\ln(\phi-a)-(\phi-a)$	$\gamma(\cdot)\cdot n$	$\phi-\exp(-\psi)$
Plasticity (simplified)	$\{\lvert\nabla u\rvert \leq \phi\}$	$-\sqrt{\phi^2-\lvert a\rvert^2}$	∇	$\frac{\phi\psi}{\sqrt{1+\lvert\psi\rvert^2}}$
Multi-phase Flows	$\{u \geq 0,\ \sum_i u_i = 1\}$	$\sum_i a_i \ln(a_i)$	id	$\frac{\exp\psi}{\sum_i \exp\psi_i}$
Monge–Ampère Equation	$\{\det(\nabla^2 u) \geq 0\}$	$\mathrm{tr}(a\ln a - a)$	∇^2	$\exp\psi$

THEORY

Energy **Principle**

$$\min_{u \in K} J(u) \quad \text{where } K := \{u \in V \mid Bu(x) \in C \text{ f.a.e. } x\}$$

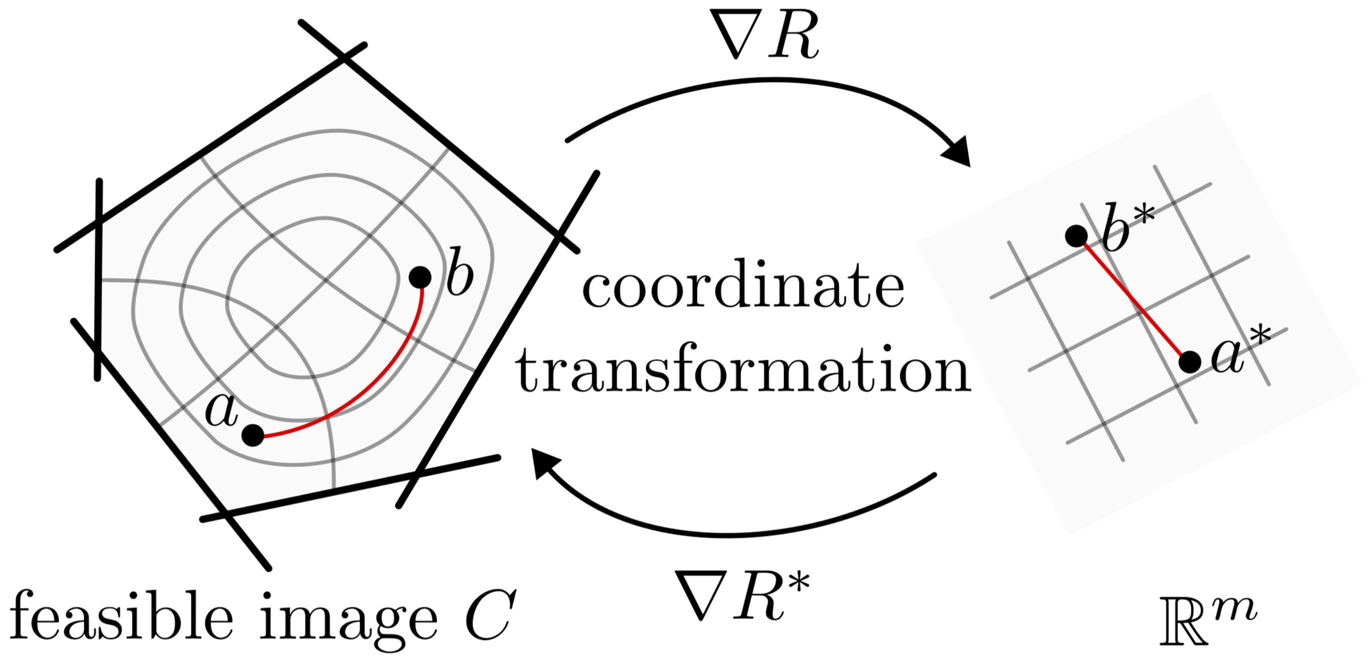
Regularized **Subproblems**

$$u^k = \arg\min_{u \in K} J(u) + {\alpha_k}^{-1} \int D_R(Bu, Bu^{k-1})\,\mathrm{d}x, \qquad k = 1, 2, \ldots$$

Euler–Lagrange **Equations**

$$\text{Find } u^k \in K: \qquad \underbrace{\alpha_k J'(u^k) + B^* \nabla R(Bu^k) - B^* \nabla R(Bu^{k-1}) = 0,}_{\text{Properties of } R \text{ force constraints to be inactive for } u^k}$$

Latent Variable **Proximal Point**



Rewrite the EL equations

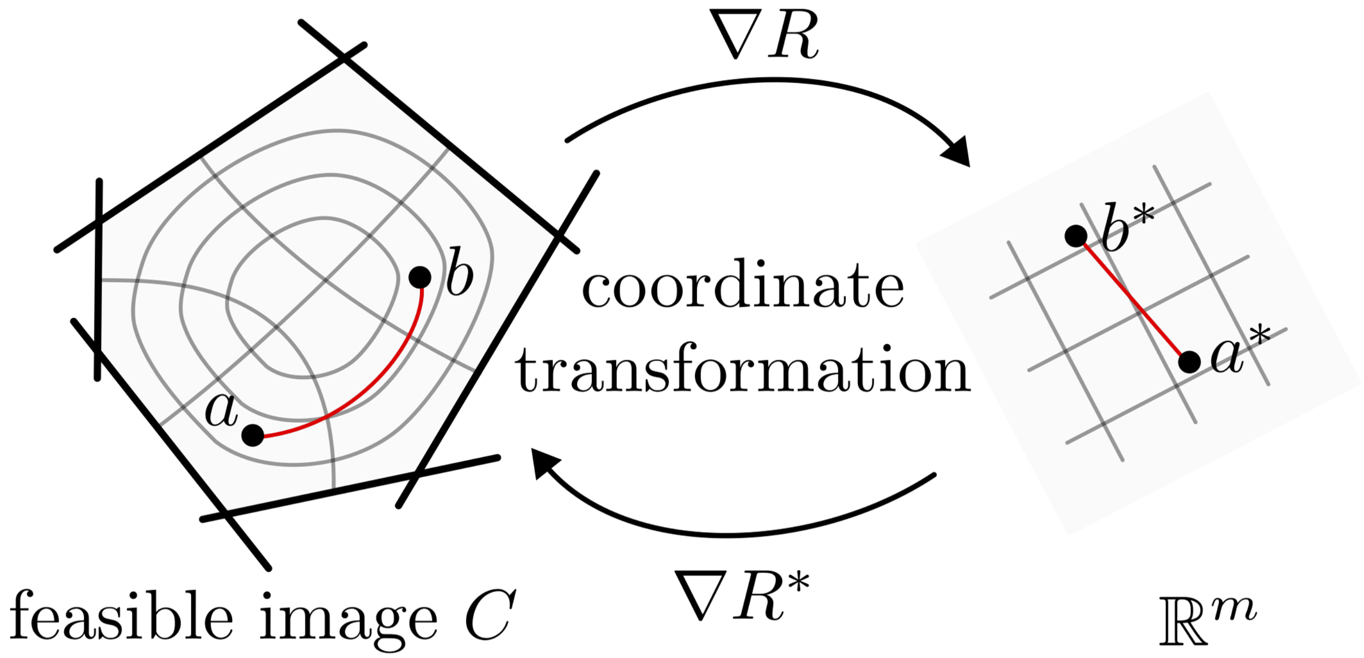
$$\alpha_k J'(u^k) + B^* \underbrace{\nabla R(Bu^k)}_{=\psi^k} - B^* \underbrace{\nabla R(Bu^{k-1})}_{=\psi^{k-1}} = 0$$

as saddle-point problems:

$$\alpha_k J'(u^k) + B^* \psi^k = B^* \psi^{k-1}$$
$$\nabla R(Bu^k) - \psi^k = 0$$

Singular Functions

Latent Variable **Proximal Point**



Rewrite the EL equations

$$\alpha_k J'(u^k) + B^* \underbrace{\nabla R(Bu^k)}_{=\psi^k} - B^* \underbrace{\nabla R(Bu^{k-1})}_{=\psi^{k-1}} = 0$$

as saddle-point problems:

$$\boxed{\begin{aligned} \alpha_k J'(u^k) + B^*\psi^k &= B^*\psi^{k-1} \\ Bu^k - \nabla R^*(\psi^k) &= 0 \end{aligned}}$$

Smooth Equations!

Unified Framework

Variational Problem with Inequality Constraints

Regularize: Sequence of nonlinear PDEs

Discretize: Nonlinear systems of equations

For each $k = 1, 2, \ldots,$ solve

$$\boxed{\begin{aligned} \alpha_k J'(u^k) + B^*\psi^k &= B^*\psi^{k-1} \\ Bu^k - \nabla R^*(\psi^k) &= 0 \end{aligned}}$$

Proximal Galerkin

=

Proximal Subproblems

+

Galerkin Discretization

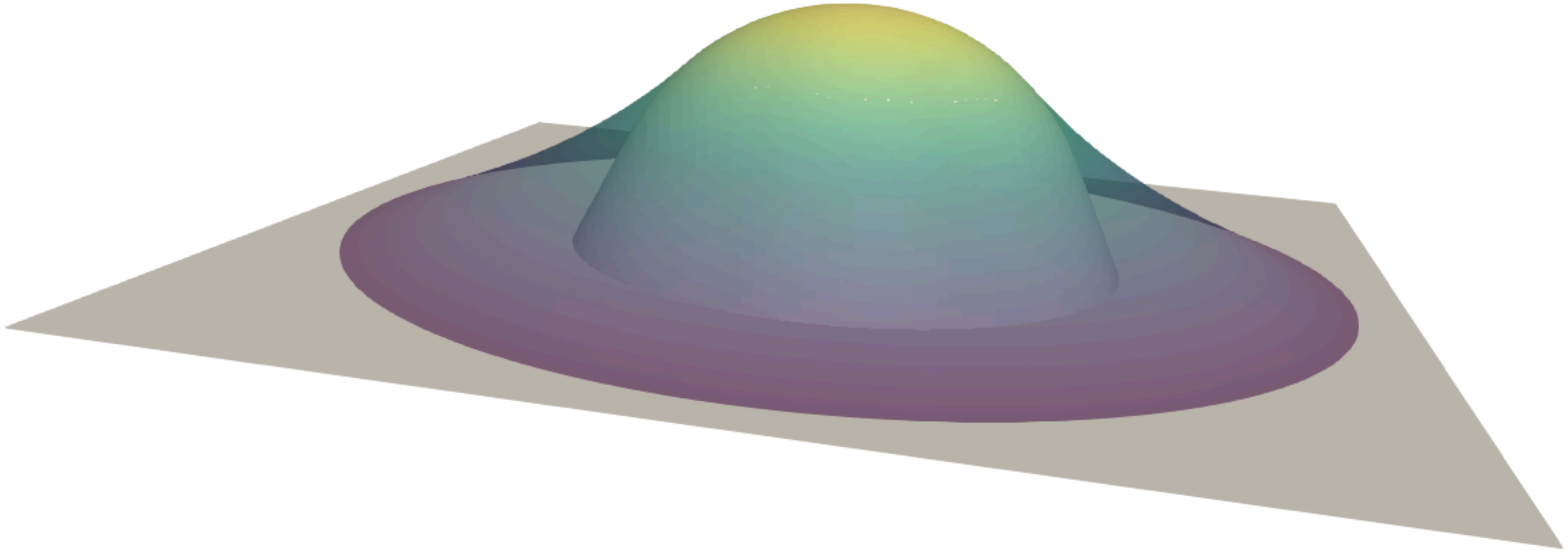
For each $k = 1, 2, \ldots,$ solve

$$\begin{aligned} \alpha_k J'(u^k) + B^* \psi^k &= B^* \psi^{k-1} \\ B u^k - \nabla R^*(\psi^k) &= 0 \end{aligned}$$

$$\begin{cases} \text{Find } u_h^k \in V_h \text{ and } \psi_h^k \in W_h \text{ such that} \\ \alpha_k \langle J'(u_h^k), v_h \rangle + (\psi_h^k, B v_h) = (\psi_h^{k-1}, B v_h) & \text{for all } v_h \in V_h \\ (B u_h^k, w_h) - (\nabla R^*(\psi_h^k), w_h) = 0 & \text{for all } w_h \in W_h \end{cases}$$

Example

$$\min_{u \in K} J(u) = \frac{1}{2}\|\nabla u\|^2 \;\text{ where } K = \{u \in H_0^1(\Omega) \mid u \geq \phi \text{ a.e. in } \Omega\}$$



Obstacle Problem: The “hello world” of contact mechanics $K = \{u \in H_0^1 \mid u \geq \phi\}$

Example

$$\min_{u\in K} J(u) = \frac{1}{2}\|\nabla u\|^2 \ \text{ where } K = \{u \in H_0^1(\Omega) \mid u \geq \phi \text{ a.e. in } \Omega\}$$

Weak Formulation

$$\alpha_k\langle J'(u^k), v\rangle + (\psi^k, Bv) = (\psi^{k-1}, Bv)$$
$$(Bu^k, w) - (\nabla R^*(\psi^k), w) = 0$$

Example

$$\min_{u \in K} J(u) = \frac{1}{2}\|\nabla u\|^2 \text{ where } K = \{u \in H_0^1(\Omega) \mid u \geq \phi \text{ a.e. in } \Omega\}$$

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$$\alpha_k \langle J'(u^k), v\rangle + (\psi^k, Bv) = (\psi^{k-1}, Bv)$$
$$(Bu^k, w) - (\nabla R^*(\psi^k), w) = 0$$

$B = \mathrm{id}$

Example

$$\min_{u \in K} J(u) = \frac{1}{2}\|\nabla u\|^2 \quad \text{where } K = \{u \in H_0^1(\Omega) \mid u \geq \phi \text{ a.e. in } \Omega\}$$

Weak Formulation

$$\alpha_k \langle J'(u^k), v\rangle + (\psi^k, v) = (\psi^{k-1}, v)$$
$$(u^k, w) - (\nabla R^*(\psi^k), w) = 0$$

$$B = \mathrm{id}$$

Example

$$\min_{u \in K} J(u) = \frac{1}{2}\|\nabla u\|^2 \quad \text{where } K = \{u \in H_0^1(\Omega) \mid u \geq \phi \text{ a.e. in } \Omega\}$$

Weak Formulation

$$\alpha_k(\nabla u^k, \nabla v) + (\psi^k, v) = (\psi^{k-1}, v)$$
$$(u^k, w) - (\nabla R^*(\psi^k), w) = 0$$

Fréchet derivative of
Dirichlet Energy

Example

$$\min_{u \in K} J(u) = \frac{1}{2}\|\nabla u\|^2 \;\; \text{where } K = \{u \in H_0^1(\Omega) \mid u \geq \phi \text{ a.e. in } \Omega\}$$

Weak Formulation

$$\alpha_k(\nabla u^k, \nabla v) + (\psi^k, v) = (\psi^{k-1}, v)$$
$$(u^k, w) - (\exp(\psi^k) + \phi, w) = 0$$

Using Shannon entropy
as Legendre function

$$R(a) = (a-\phi)\ln(a-\phi) - (a-\phi)$$
$$\implies R^*(a^*) = \exp(a^*) + \phi a^*$$

Example

$$\min_{u \in K} J(u) = \frac{1}{2}\|\nabla u\|^2 \;\text{ where } K = \{u \in H_0^1(\Omega) \mid u \geq \phi \text{ a.e. in } \Omega\}$$

Weak Formulation

$$\alpha_k(\nabla u^k, \nabla v) + (\psi^k, v) = (\psi^{k-1}, v)$$
$$(u^k, w) - (\exp(\psi^k) + \phi, w) = 0$$

Galerkin discretization

$$u_h \in V_h \;\text{ and }\; \tilde{u}_h = \phi + \exp \psi_h$$

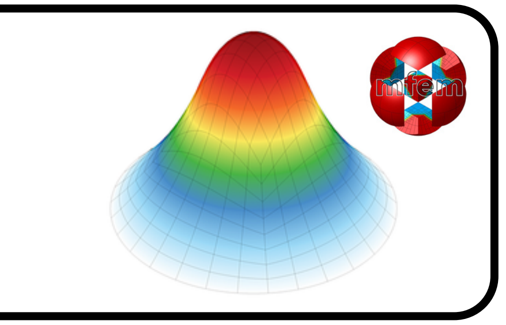
Second solution **always preserves constraints**

Requires **iterating** a non-linear map

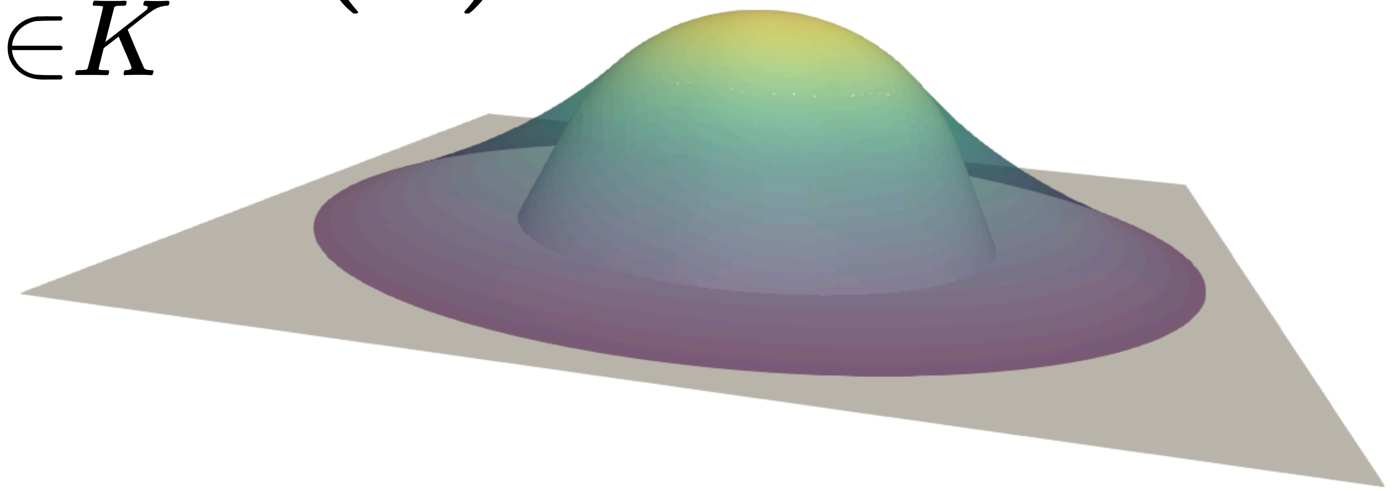
Does not require sending any parameter to zero/infinty

Method Comparison

See also: MFEM Example 36



$$\min_{u \in K} J(u)$$



Obstacle Problem: The "hello world" of contact mechanics $K = \{u \in H_0^1 \mid u \geq \phi\}$

	Degree $p = 1$			Degree $p = 2$		
Method	h	$h/2$	$h/4$	h	$h/2$	$h/4$
PG	9	9	9	10	10	9
AS [21]	8	11	17	Not high order bound preserving		
AL [22]	4	7	12			
IP [23]	8	8	7			

Number of iterations for convergence of standard solvers

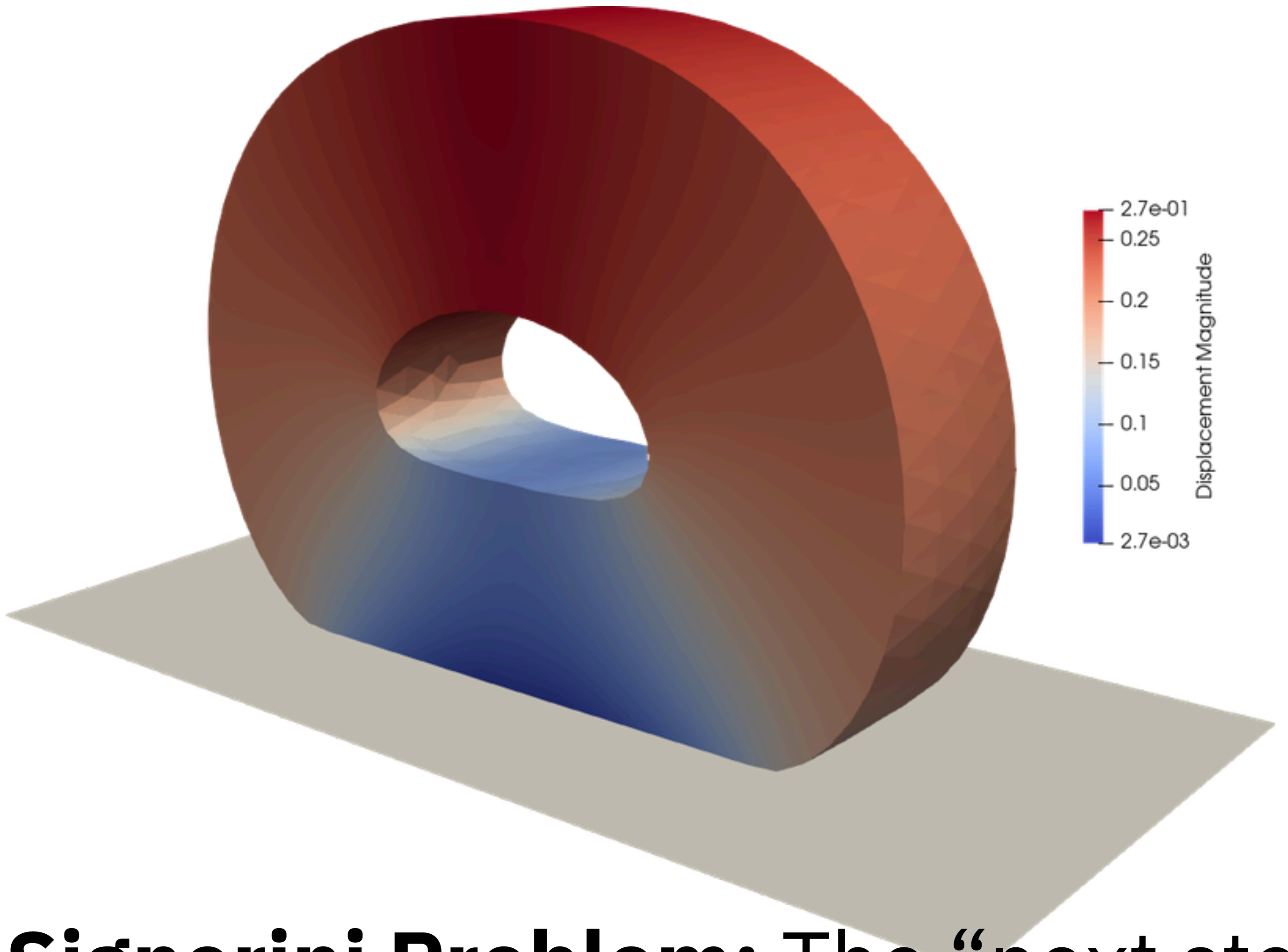
Only the **"geometric methods"** (Proximal Galerkin and Interior Point) are mesh-**independent**

Only Proximal Galerkin is **bound preserving** *everywhere*

CLICK FOR THEORY

Example

$$\min_{u \in K} J(u) \quad \text{where } K = \{u \in V \subset H^1(\Omega)^3 \mid u \cdot n \geq \phi \text{ a.e. on } \partial\Omega\}$$



Signorini Problem: The “next step” towards contact mechanics

Example

$$\min_{u \in K} J(u) \quad \text{where } K = \{u \in V \subset H^1(\Omega)^3 \mid u \cdot n \geq \phi \text{ a.e. on } \partial\Omega\}$$

Weak Formulation

$$\alpha_k \langle J'(u^k), v\rangle + (\psi^k, Bv)_{\partial\Omega} = (\psi^{k-1}, Bv)_{\partial\Omega}$$
$$(Bu^k, w)_{\partial\Omega} - (\nabla R^*(\psi^k), w)_{\partial\Omega} = 0$$

$$B = \gamma(\cdot) \cdot n$$

Example

$$\min_{u \in K} J(u) \;\; \text{where } K = \{u \in V \subset H^1(\Omega)^3 \mid u \cdot n \geq \phi \text{ a.e. on } \partial\Omega\}$$

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$$\alpha_k \langle J'(u^k), v\rangle + (\psi^k, v \cdot n)_{\partial\Omega} = (\psi^{k-1}, v \cdot n)_{\partial\Omega}$$
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Using Shannon entropy as Legendre function

$$R(a) = (a-\phi)\ln(a-\phi) - (a-\phi)$$
$$\implies R^*(a^*) = \exp(a^*) + \phi a^*$$

Example

$$\min_{u \in K} J(u) \quad \text{where } K = \{u \in V \subset H^1(\Omega)^3 \mid u \cdot n \geq \phi \text{ a.e. on } \partial\Omega\}$$

Weak Formulation

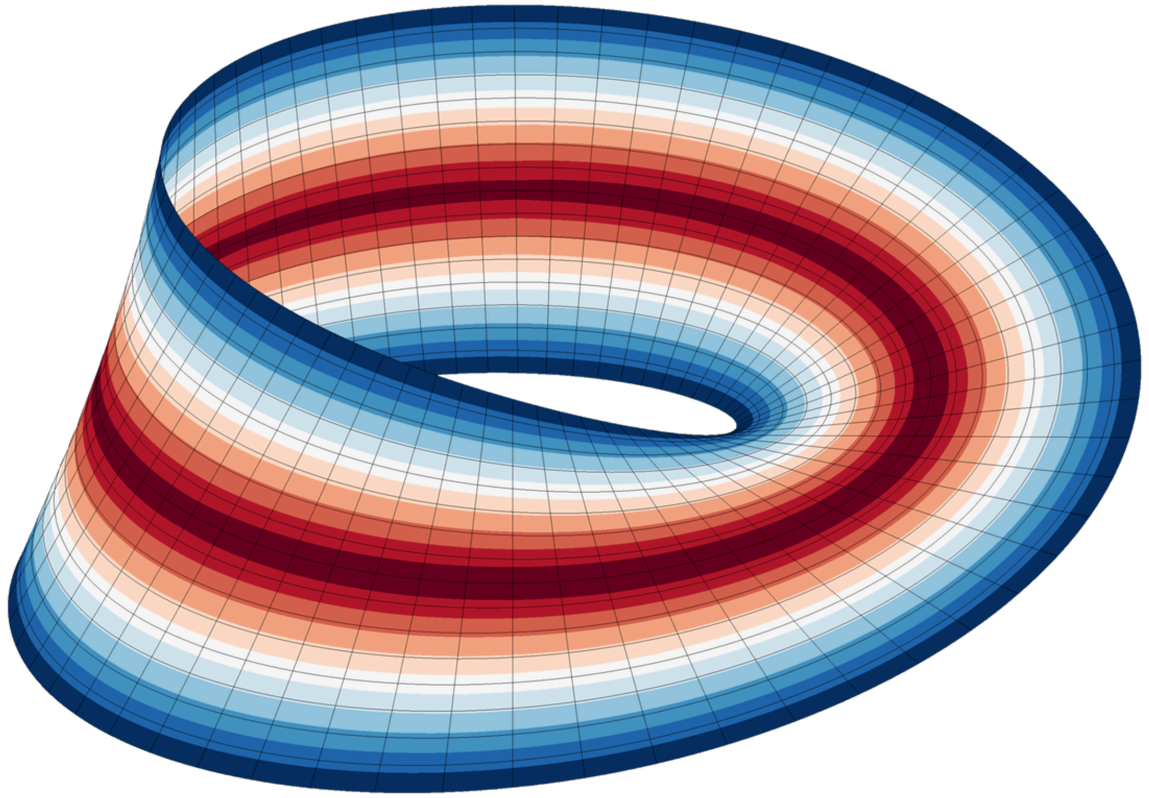
$$\alpha_k \langle J'(u^k), v \rangle + (\psi^k, v \cdot n)_{\partial\Omega} = (\psi^{k-1}, v \cdot n)_{\partial\Omega}$$
$$(u^k \cdot n, w)_{\partial\Omega} - (\exp(\psi^k), w)_{\partial\Omega} = (\phi, w)_{\partial\Omega}$$

Using Shannon entropy as Legendre function

$$R(a) = (a - \phi)\ln(a - \phi) - (a - \phi)$$
$$\implies R^*(a^*) = \exp(a^*) + \phi a^*$$

Example

$$\min_{u \in K} - \int_\Omega u \,\mathrm{d}x \quad \text{where } K = \{u \in W_0^{1,\infty}(\Omega) \mid |\nabla u| \leq 1 \text{ a.e. in } \Omega\}$$



Vicosity solutions to the **eikonal equation:**

$$|\nabla u| = 1 \ \text{ in } \Omega, \qquad u = 0 \ \text{ on } \partial\Omega$$

Example

$$\min_{u \in K} - \int_\Omega u \, \mathrm{d}x \quad \text{where } K = \{u \in W_0^{1,\infty}(\Omega) \mid |\nabla u| \leq 1 \text{ a.e. in } \Omega\}$$

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$$(Bu^k, w) - (\nabla R^*(\psi^k), w) = 0$$

$$B = \nabla$$

Example

$$\min_{u \in K} - \int_\Omega u \, \mathrm{d}x \quad \text{where } K = \{u \in W_0^{1,\infty}(\Omega) \mid |\nabla u| \leq 1 \text{ a.e. in } \Omega\}$$

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$$\alpha_k \langle J'(u^k), v \rangle + (\psi^k, \nabla v) = (\psi^{k-1}, \nabla v)$$
$$(\nabla u^k, w) - (\nabla R^*(\psi^k), w) = 0$$

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Example

$$\min_{u \in K} -\int_\Omega u \,\mathrm{d}x \quad \text{where } K = \{u \in W_0^{1,\infty}(\Omega) \mid |\nabla u| \leq 1 \text{ a.e. in } \Omega\}$$

Weak Formulation

$$-\alpha_k(1, v) + (\psi^k, \nabla v) = (\psi^{k-1}, \nabla v)$$
$$(\nabla u^k, w) - (\nabla R^*(\psi^k), w) = 0$$

Objective function
is linear

Example

$$\min_{u\in K} -\int_\Omega u \,\mathrm{d}x \quad \text{where } K = \{u \in W_0^{1,\infty}(\Omega) \mid |\nabla u| \leq 1 \text{ a.e. in } \Omega\}$$

Weak Formulation

$$-\alpha_k(1, v) + (\psi^k, \nabla v) = (\psi^{k-1}, \nabla v)$$
$$(\nabla u^k, w) - (\nabla R^*(\psi^k), w) = 0$$

Using *Hellinger entropy* as Legendre function

$$R(a) = -\sqrt{1 - |a|^2}$$

$$\implies R^*(a^*) = \sqrt{1 + |a^*|^2}$$

Example

$$\min_{u\in K} -\int_\Omega u \,\mathrm{d}x \quad \text{where } K = \{u \in W_0^{1,\infty}(\Omega) \mid |\nabla u| \leq 1 \text{ a.e. in } \Omega\}$$

Weak Formulation

$$-\alpha_k(1, v) + (\psi^k, \nabla v) = (\psi^{k-1}, \nabla v)$$

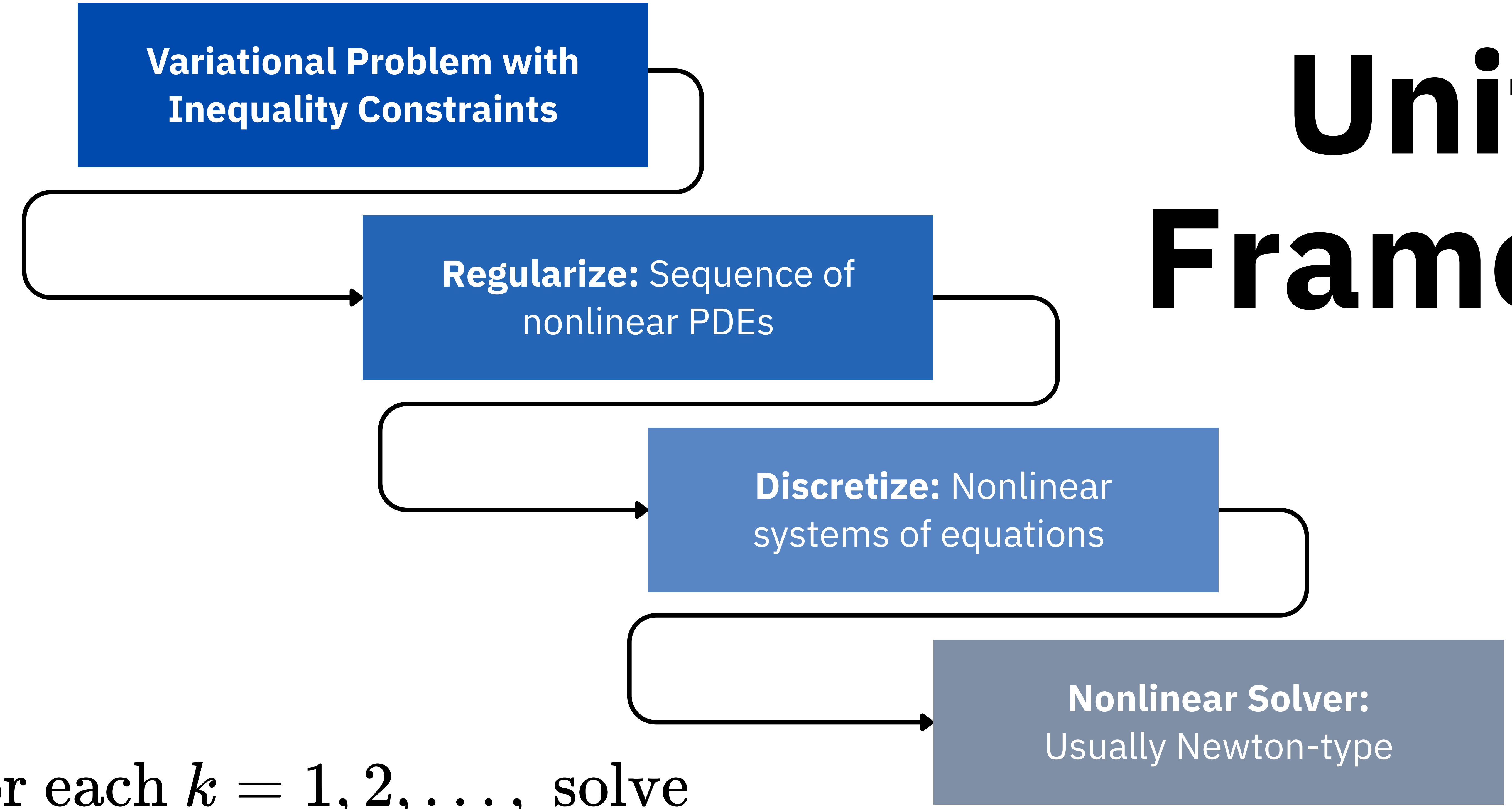
$$(\nabla u^k, w) - \left(\frac{\psi^k}{\sqrt{1+|\psi^k|^2}}, w\right) = 0$$

$$R(a) = -\sqrt{1-|a|^2}$$

$$\implies R^*(a^*) = \sqrt{1+|a^*|^2}$$

Using *Hellinger entropy* as Legendre function

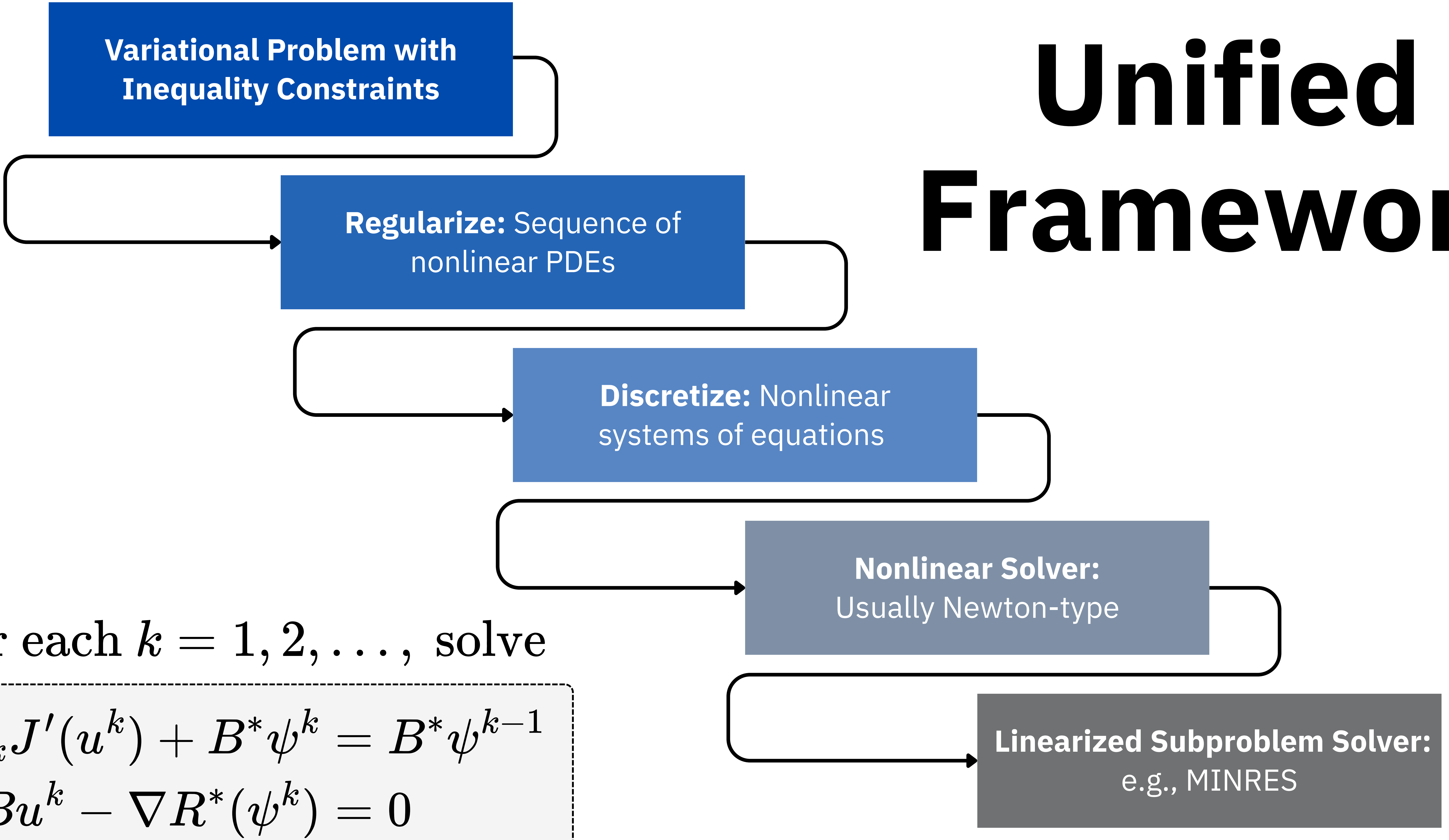
Unified Framework



For each $k = 1, 2, \ldots,$ solve

$$\begin{aligned} \alpha_k J'(u^k) + B^* \psi^k &= B^* \psi^{k-1} \\ B u^k - \nabla R^*(\psi^k) &= 0 \end{aligned}$$

Unified Framework



For each $k = 1, 2, \ldots,$ solve

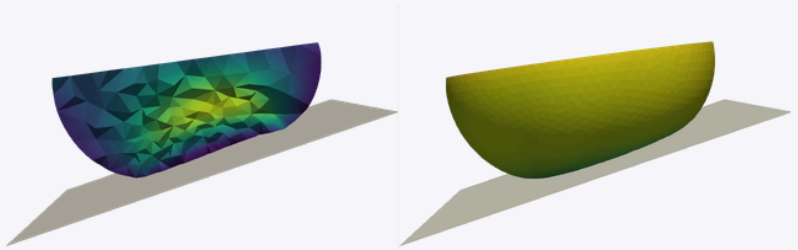
$$\begin{aligned} \alpha_k J'(u^k) + B^*\psi^k &= B^*\psi^{k-1} \\ Bu^k - \nabla R^*(\psi^k) &= 0 \end{aligned}$$

Applications

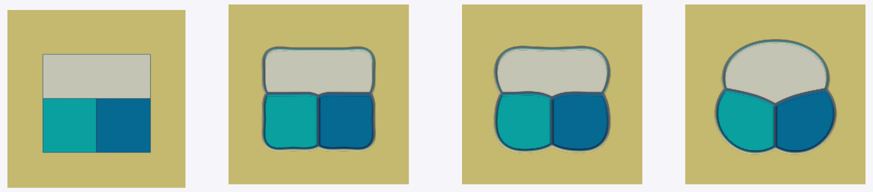
References

1. **Keith**, B., & **Surowiec**, T. M. (2024). *Found. Comput. Math.* **Winner of the Langtangen Prize!**
2. **Keith**, B., **Kim**, D., **Lazarov**, B. S., & **Surowiec**, T. M. (2025). *SIAM J. Opt.*
3. **Kim**, D., **Lazarov**, B. S., **Surowiec**, T. M., & **Keith**, B. (2025). Struct. Mult. Disp. Opt.
4. **Dokken**, J.S., **Farrell**, P.E., **Keith**, B., **Papadopoulos**, I., **Surowiec**, T.M. (2025). CMAME
5. **Keith**, B., **Masri**, R., **Zeinhofer**, M. (2025). arXiv:2507.13516
6. **Fu**, G., **Keith**, B., & **Masri**, R. (2026). *Math. Comp.*
7. **Fu**, G., **Keith**, B., **Kim**, D., **Masri**, R., & **Pazner**, W. (2026). arXiv:2602.14967

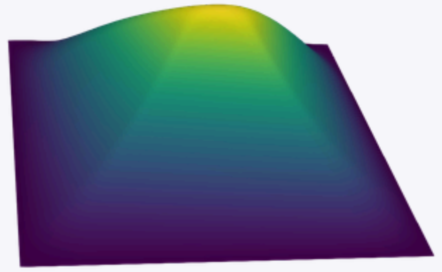
FEniCSx



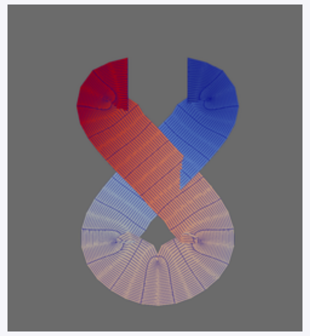
Contact



Multi-Phase Flows

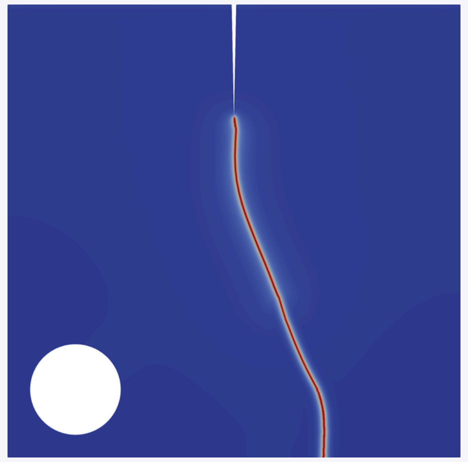


Elasto-Plasticity

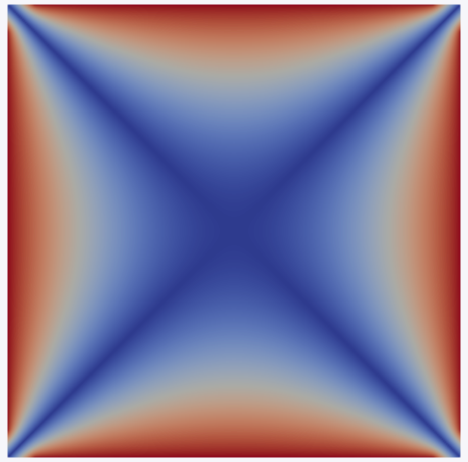


Large-deformation Elasticity

FireDrake

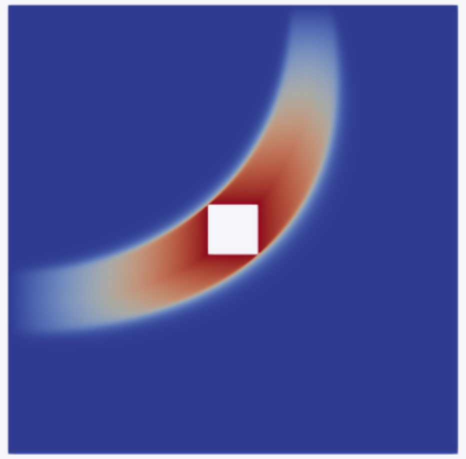


Variational Fracture



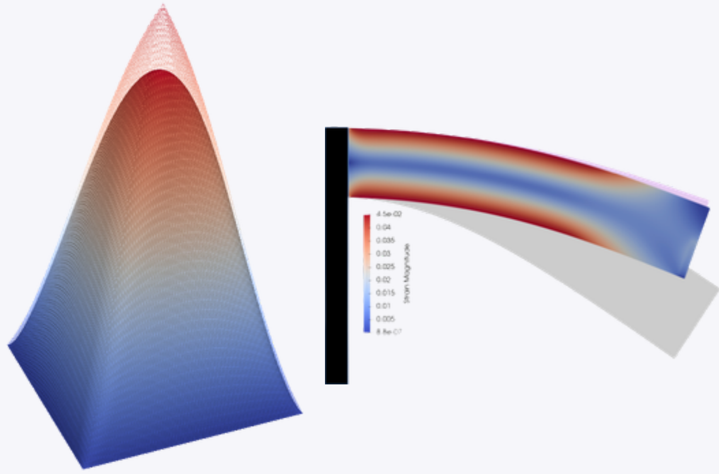
Liquid Crystals

NGSolve



Anisotropic Diffusion

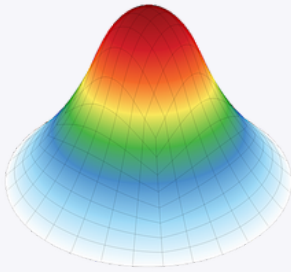
GridAp



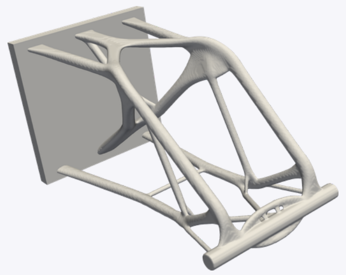
Quasi-Variational Ineqs.

Official MFEM Examples

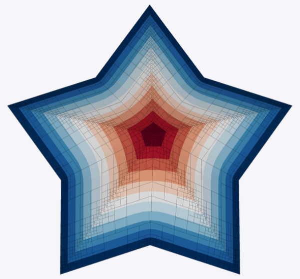
Example 36 (Obstacle prob.)



Example 37 (Topology opt.)



Example 40 (Eikonal eq.)



THEORY OVERVIEW

Joint work with

Postdoc
R. Masri

Postdoc
M. Zeinhofer

Scan Me!

The objective function **decreases** globally:

$$J(u_h^k) \leq J(u_h^{k-1}) \ \text{ for all } k \geq 1, \ h > 0$$

THEORY OVERVIEW

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Scan Me!

The objective function **decreases** globally:

$$J(u_h^k) \leq J(u_h^{k-1}) \ \text{ for all } k \geq 1, \ h > 0$$

Proof:

$$J(u_h^k) \leq J(u_h^{k-1}) + \langle J'(u_h^k), u_h^k - u_h^{k-1} \rangle$$

Subgradient Inequality

THEORY OVERVIEW

Joint work with

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R. Masri

Postdoc
M. Zeinhofer

Scan Me!

The objective function **decreases** globally:

$$J(u_h^k) \leq J(u_h^{k-1}) \quad \text{for all } k \geq 1, \;\; h > 0$$

$$\begin{cases} \text{Find } u_h^k \in V_h \text{ and } \psi_h^k \in W_h \text{ such that} & \\ \alpha_k \langle J'(u_h^k), v_h \rangle + (\psi_h^k, Bv_h) = (\psi_h^{k-1}, Bv_h) & \text{for all } v_h \in V_h \\ (Bu_h^k, w_h) - (\nabla R^*(\psi_h^k), w_h) = 0 & \text{for all } w_h \in W_h \end{cases}$$

Recall: Proximal Galerkin subproblems

Proof:

$$J(u_h^k) \leq J(u_h^{k-1}) + \langle J'(u_h^k), u_h^k - u_h^{k-1} \rangle$$

THEORY OVERVIEW

Joint work with

Postdoc
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Scan Me!

The objective function **decreases** globally:

$$J(u_h^k) \le J(u_h^{k-1}) \ \text{ for all } k \ge 1,\ h > 0$$

$$\begin{cases} \text{Find } u_h^k \in V_h \text{ and } \psi_h^k \in W_h \text{ such that} & \\ \alpha_k \langle J'(u_h^k), v_h \rangle + (\psi_h^k, B v_h) = (\psi_h^{k-1}, B v_h) & \text{for all } v_h \in V_h \\ (B u_h^k, w_h) - (\nabla R^*(\psi_h^k), w_h) = 0 & \text{for all } w_h \in W_h \end{cases}$$

Proof:

$$J(u_h^k) \le J(u_h^{k-1}) + \langle J'(u_h^k), u_h^k - u_h^{k-1} \rangle$$

Set $v_h = u_h^k - u_h^{k-1}$

THEORY OVERVIEW

Joint work with

Postdoc
R. Masri

Postdoc
M. Zeinhofer

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The objective function **decreases** globally:

$$J(u_h^k) \leq J(u_h^{k-1}) \;\; \text{for all } k \geq 1, \;\; h > 0$$

Proof:

$$\begin{cases} \text{Find } u_h^k \in V_h \text{ and } \psi_h^k \in W_h \text{ such that} \\ \alpha_k \langle J'(u_h^k), v_h \rangle + (\psi_h^k, Bv_h) = (\psi_h^{k-1}, Bv_h) & \text{for all } v_h \in V_h \\ (Bu_h^k, w_h) - (\nabla R^*(\psi_h^k), w_h) = 0 & \text{for all } w_h \in W_h \end{cases}$$

Set $v_h = u_h^k - u_h^{k-1}$

$$\begin{aligned} J(u_h^k) &\leq J(u_h^{k-1}) + \langle J'(u_h^k), u_h^k - u_h^{k-1} \rangle \\ &= J(u_h^{k-1}) - \frac{1}{\alpha_k}(Bu_h^{k-1} - Bu_h^k, \psi_h^{k-1} - \psi_h^k) \end{aligned}$$

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The objective function **decreases** globally:

$$J(u_h^k) \leq J(u_h^{k-1}) \;\text{ for all } k \geq 1,\; h > 0$$

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Proof:

$$\begin{aligned} J(u_h^k) &\leq J(u_h^{k-1}) + \langle J'(u_h^k), u_h^k - u_h^{k-1} \rangle \\ &= J(u_h^{k-1}) - \frac{1}{\alpha_k}(Bu_h^{k-1} - Bu_h^k, \psi_h^{k-1} - \psi_h^k) \\ &= J(u_h^{k-1}) - \frac{1}{\alpha_k}(\nabla R^*(\psi_h^k) - \nabla R^*(\psi_h^{k-1}), \psi_h^k - \psi_h^{k-1}) \end{aligned}$$

Set $w_h = \psi_h^k - \psi_h^{k-1}$

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The objective function **decreases** globally:

$$J(u_h^k) \leq J(u_h^{k-1}) \ \text{ for all } k \geq 1, \ h > 0$$

Proof:

$$\begin{aligned} J(u_h^k) &\leq J(u_h^{k-1}) + \langle J'(u_h^k), u_h^k - u_h^{k-1} \rangle \\ &= J(u_h^{k-1}) - \frac{1}{\alpha_k}(Bu_h^{k-1} - Bu_h^k, \psi_h^{k-1} - \psi_h^k) \\ &= J(u_h^{k-1}) - \frac{1}{\alpha_k}(\nabla R^*(\psi_h^k) - \nabla R^*(\psi_h^{k-1}), \psi_h^k - \psi_h^{k-1}) \end{aligned}$$

(By Monotonicity)

≥ 0

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The objective function **decreases** globally:

$$J(u_h^k) \leq J(u_h^{k-1}) \ \text{ for all } k \geq 1, \ h > 0$$

Independent optimization and discretization error terms:

Approximation Error

$$\|u^\star - u_h^k\|_V^2 \leq \frac{C_{\mathrm{stab}}}{\sum_{i=1}^k \alpha_i} + C_{\mathrm{reg}} h^{2s}$$

Discretization Error

Optimization Error

Note: The proof uses that the objective function is *quadratic* and *coercive*

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The objective function **decreases** globally:

$$J(u_h^k) \leq J(u_h^{k-1}) \ \text{ for all } k \geq 1, \ h > 0$$

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$$\|u^\star - u_h^k\|_V^2 \leq \frac{C_{\text{stab}}}{\sum_{i=1}^k \alpha_i} + C_{\text{reg}} h^{2s}$$

Discretization Error

Optimization Error

Avoids singular parameters

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The objective function **decreases** globally:

$$J(u_h^k) \leq J(u_h^{k-1}) \ \text{ for all } k \geq 1, \ h > 0$$

Independent optimization and discretization error terms:

Approximation Error

$$\|u^\star - u_h^k\|_V^2 \leq \frac{C_{\text{stab}}}{\sum_{i=1}^k \alpha_i} + C_{\text{reg}} h^{2s}$$

Discretization Error

Optimization Error

Independent error terms imply **mesh-independence!**

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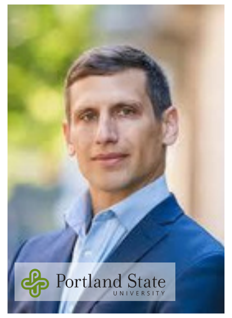

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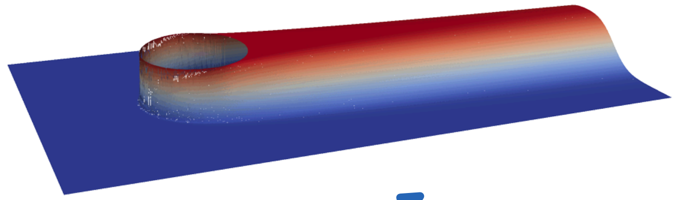
W. Pazner



Bound-preserving convection-diffusion

Inequality constraints **without** an energy principle:

$$a(u^\star, v - u^\star) \geq F(v - u^\star) \quad \text{for all } v \in K$$

where the bilinear form is **non-symmetric**

$$a(u, v) \neq a(v, u)$$

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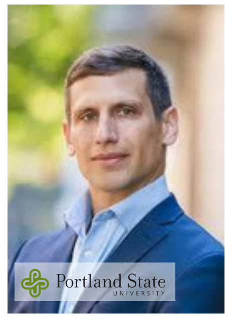
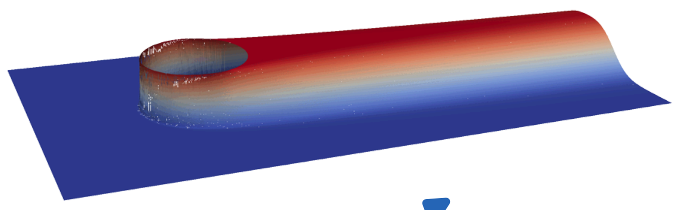
D. Kim

Asst. Professor
G. Fu



Asst. Professor
W. Pazner



Inequality constraints **without** an energy principle:

$$a(u^\star, v - u^\star) \geq F(v - u^\star) \;\; \text{for all } v \in K$$

where the bilinear form is **non-symmetric**

$$a(u, v) \neq a(v, u)$$

Bound-preserving convection-diffusion

Applications in finance and fluid mechanics

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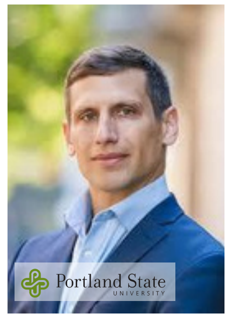
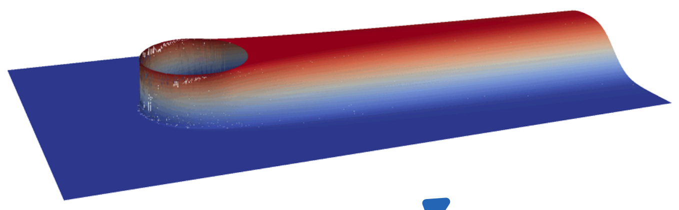


Asst. Professor
W. Pazner



Inequality constraints **without** an energy principle:

$$a(u^\star, v - u^\star) \geq F(v - u^\star) \quad \text{for all } v \in K$$

Bound-preserving convection-diffusion

where the bilinear form is **non-symmetric**

$$a(u, v) \neq a(v, u)$$

Also works for strongly Monotone operators

Algorithm:

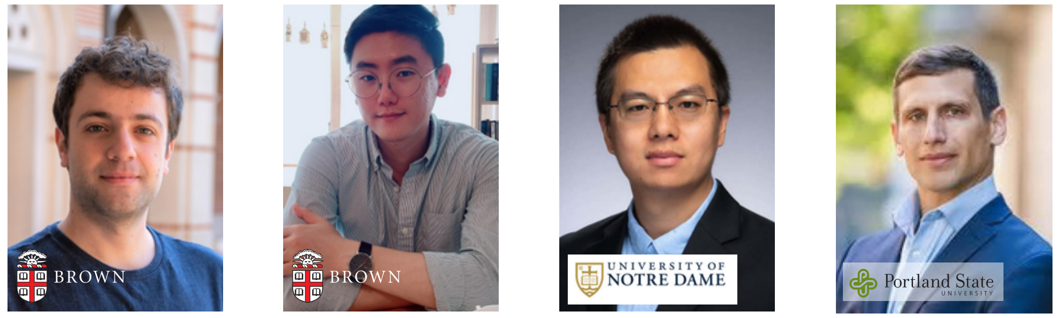
For each $k = 1, 2, \ldots,$ solve

$$\boxed{\begin{aligned} \alpha_k A(u^k) + B^*\psi^k &= B^*\psi^{k-1} \\ Bu^k - \nabla R^*(\psi^k) &= 0 \end{aligned}}$$

where $A(u) = a(u, \cdot)$

THEORY OVERVIEW

Joint work with



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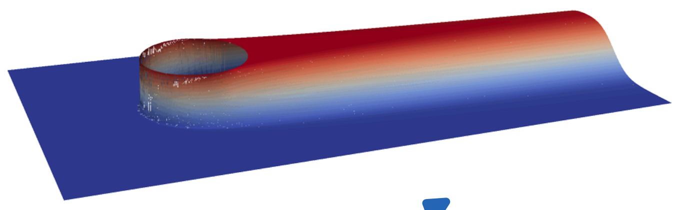


Bound-preserving convection-diffusion

Inequality constraints **without** an energy principle:

$$a(u^\star, v - u^\star) \geq F(v - u^\star) \ \text{ for all } v \in K$$

where the bilinear form is **non-symmetric**

$$a(u, v) \neq a(v, u)$$

Can prove convergence of the **average** iterate:

$$\|u^\star - \overline{u}_h^k\|_V^2 \leq \frac{C_{\text{stab}}}{\sum_{i=1}^k \alpha_i} + C_{\text{reg}} h^{2s}, \qquad \overline{u}_h^k := \frac{\sum_{\ell=1}^k \alpha^\ell u_h^\ell}{\sum_{\ell=1}^k \alpha^\ell}$$

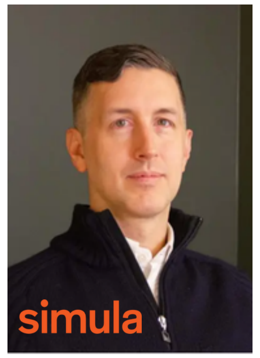
Feasible Sets in **Mechanics**

$$K = \{v \in V \mid Bv(x) \in C(x) \text{ for almost every } x \text{ in a subset of } \overline{\Omega}\}.$$

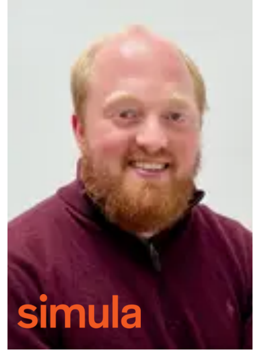
	Feasible set K	Legendre function R	B	$\nabla R^*(\psi)$
Obstacle Problems	$\{u \geq \phi\}$	$(a-\phi)\ln(a-\phi)-(a-\phi)$	id	$\phi + \exp\psi$
Topology Optimization **Variational Fracture**	$\{\phi_1 \leq u \leq \phi_2\}$	$(a-\phi_1)\ln(a-\phi_1)+(\phi_2-a)\ln(\phi_2-a)$	id	$\dfrac{\phi_1+\phi_2\exp\psi}{1+\exp\psi}$
Contact	$\{\gamma u \geq \phi\}$	$(a-\phi)\ln(a-\phi)-(a-\phi)$	γ	$\phi + \exp\psi$
	$\{(\gamma u)\cdot n \leq \phi\}$	$(\phi-a)\ln(\phi-a)-(\phi-a)$	$\gamma(\cdot)\cdot n$	$\phi - \exp(-\psi)$
Plasticity (simplified)	$\{\lvert\nabla u\rvert \leq \phi\}$	$-\sqrt{\phi^2-\lvert a\rvert^2}$	∇	$\dfrac{\phi\psi}{\sqrt{1+\lvert\psi\rvert^2}}$
Multi-phase Flows	$\{u \geq 0,\ \sum_i u_i = 1\}$	$\sum_i a_i \ln(a_i)$	id	$\dfrac{\exp\psi}{\sum_i \exp\psi_i}$
Monge–Ampère Equation	$\{\det(\nabla^2 u) \geq 0\}$	$\mathrm{tr}(a\ln a - a)$	∇^2	$\exp\psi$

Unilateral **Constraints**

Joint work with



Chief Research Scientist
T.M. Surowiec

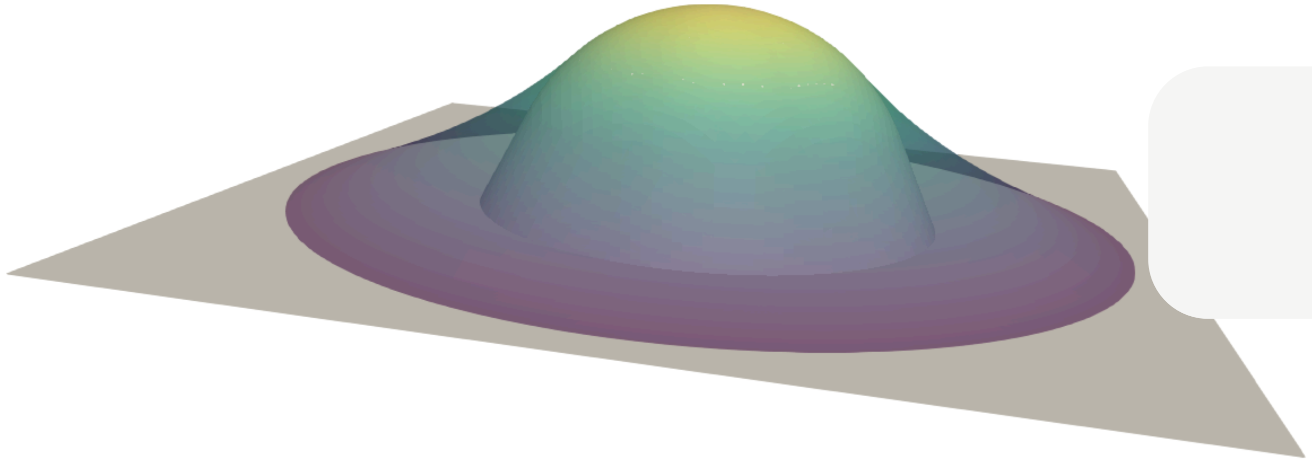


Sr. Research Engineer
J.S. Dokken

Feasible Set

$$K = \{Bu \geq \phi\}$$

Obstacle Problems

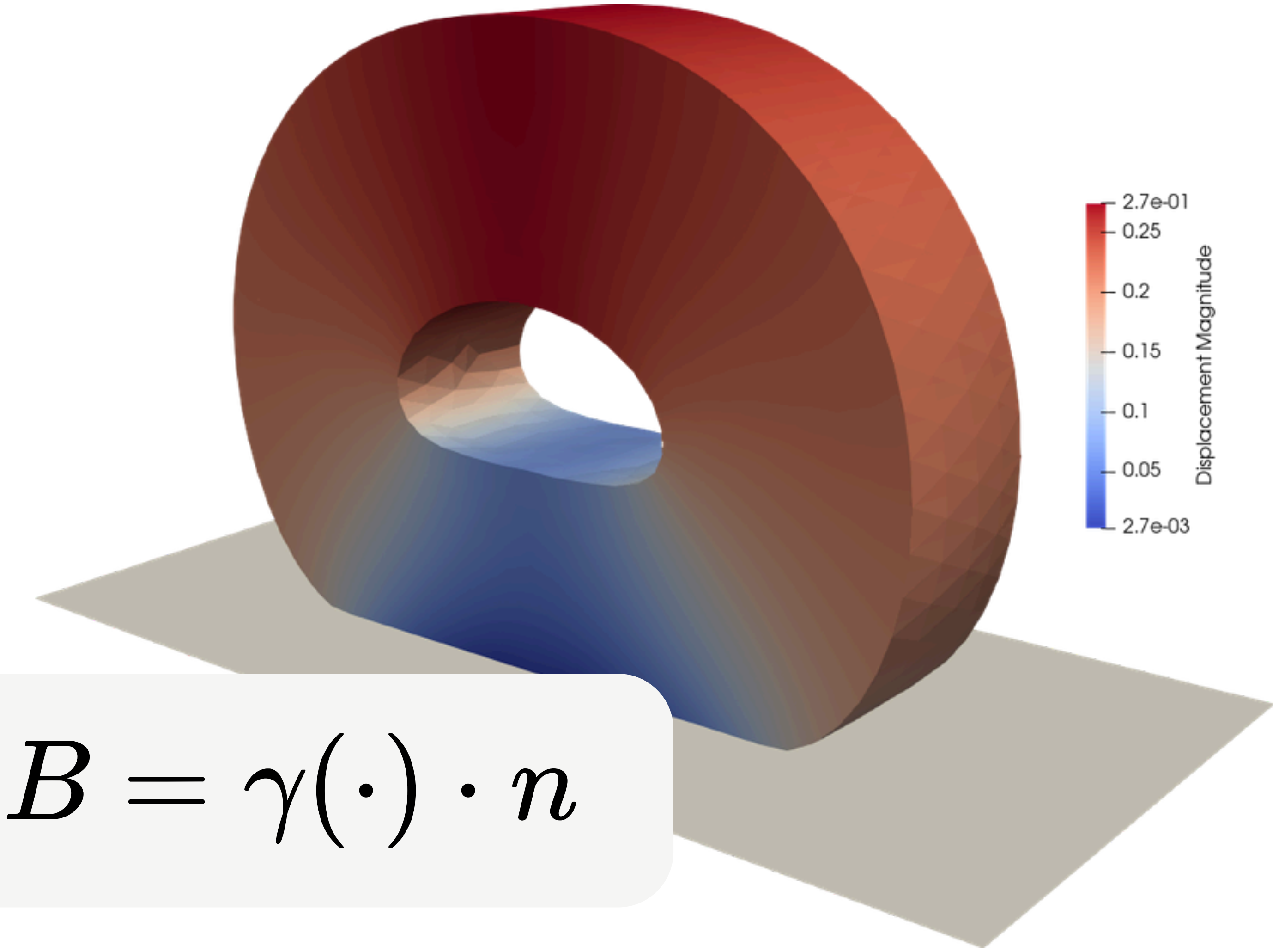


$$B = \mathrm{id}$$

For each $k = 1, 2, \ldots,$ solve

$$\begin{aligned} \alpha_k J'(u^k) + B^*\psi^k &= B^*\psi^{k-1} \\ Bu^k - \exp(\psi^k) &= 0 \end{aligned}$$

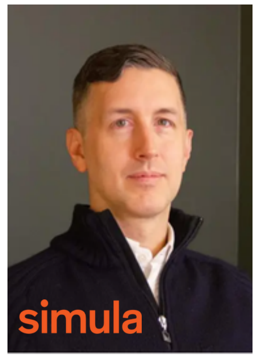
Contact Problems



$$B = \gamma(\cdot) \cdot n$$

Box **Constraints**

Joint work with



Chief Research Scientist
T.M. Surowiec

Professor
P.E. Farrell

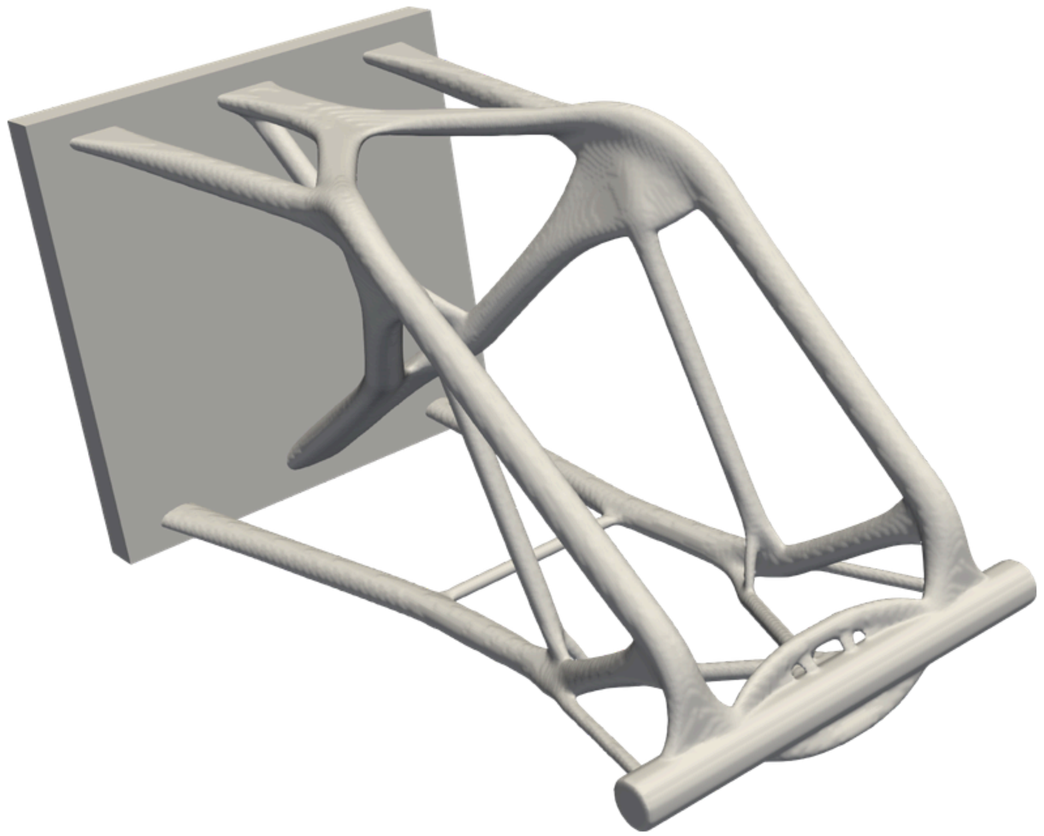
Postdoc
D. Kim

Research Engineer
B. Lazarov

Postdoc
B. Khara

Professor
Y. Bazilevs

...and others!



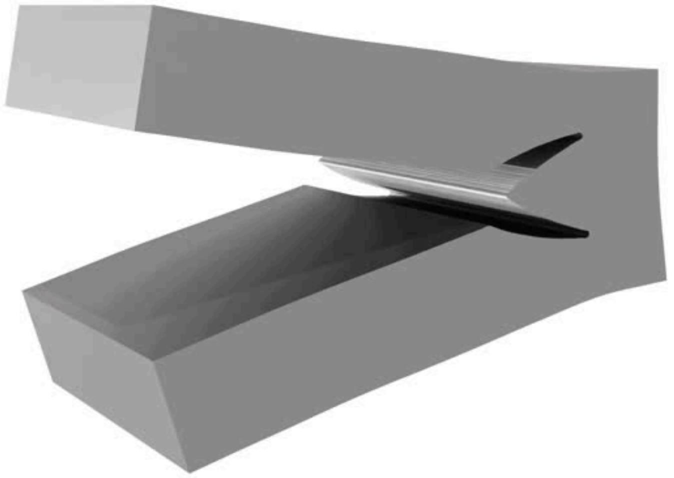
The **SiMPL method** for topology optimization

Feasible Set

$$K = \{\phi_1 \leq \rho \leq \phi_2\}$$

Coordinate Transf.

$$\nabla R^*(\psi) = \frac{\phi_1 + \phi_2 \exp \psi}{1 + \exp \psi}$$



Variational **Fracture**

Scan Me!

[1] Kim, D., Lazarov, B. S., Surowiec, T. M., & Keith, B. (2025). A Simple Introduction to the SiMPL Method for Density-Based Topology Optimization. Structural and Multidisciplinary Optimization, 68 (4), 1-17
[2] Keith, B., Kim, D., Lazarov, B. S., & Surowiec, T. M. (2025). Analysis of the SiMPL method for density-based topology optimization. SIAM Journal on Optimization, 35 (2), 1134-1164

Other **Constraints**

Applications

- Obstacle Problem
- Contact Mechanics
- Variational Fracture
- Multi-Phase Flows
- Plasticity
- Eikonal Equation
- Monge–Ampère Equation
- Topology Optimization

...and more!

Scan Me!

Gradient Constraints

$$K = \{|\nabla u| \leq \phi\}$$

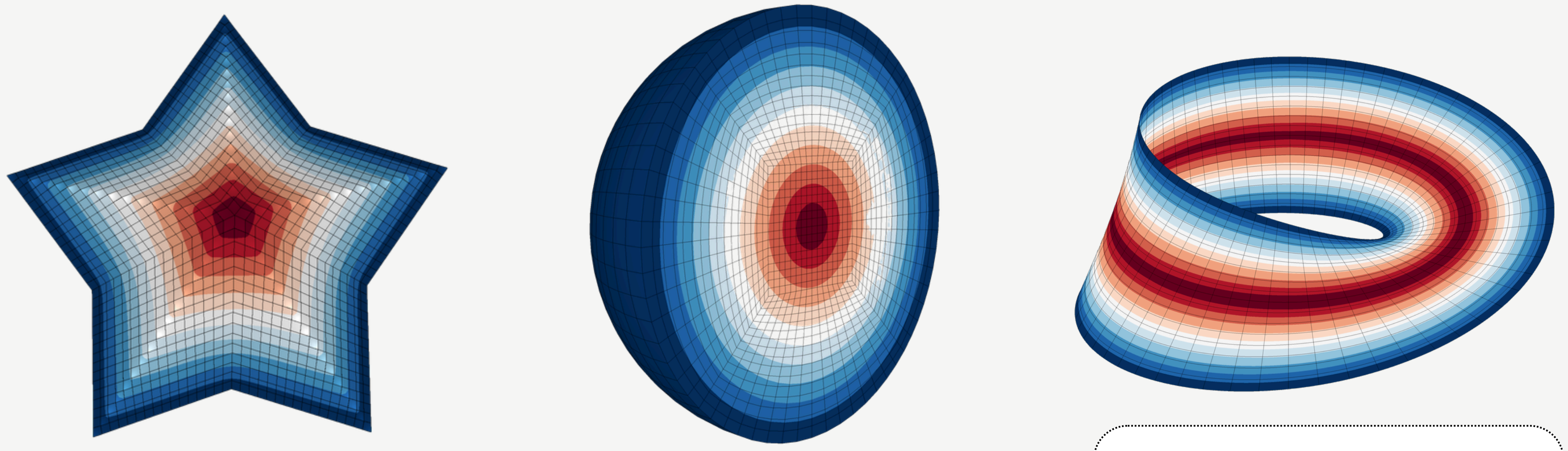
Elasto-Plastic Torsion
& Eikonal Equation

Eikonal Equation

$$J(u) := -\int_\Omega u \, \mathrm{d}x$$

Elasto-Plastic Torsion

$$J(u) := \frac{1}{2} \int_\Omega |\nabla u|^2 \, \mathrm{d}x$$



Viscosity solutions to eikonal eqn.

$$|\nabla u| = 1 \text{ in } \Omega$$
$$u = 0 \text{ on } \partial\Omega$$

Other **Constraints**

Applications

- Obstacle Problem
- Contact Mechanics
- Variational Fracture
- Multi-Phase Flows
- Plasticity
- Eikonal Equation
- Monge–Ampère Equation
- Topology Optimization

...and more!

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Gradient Constraints

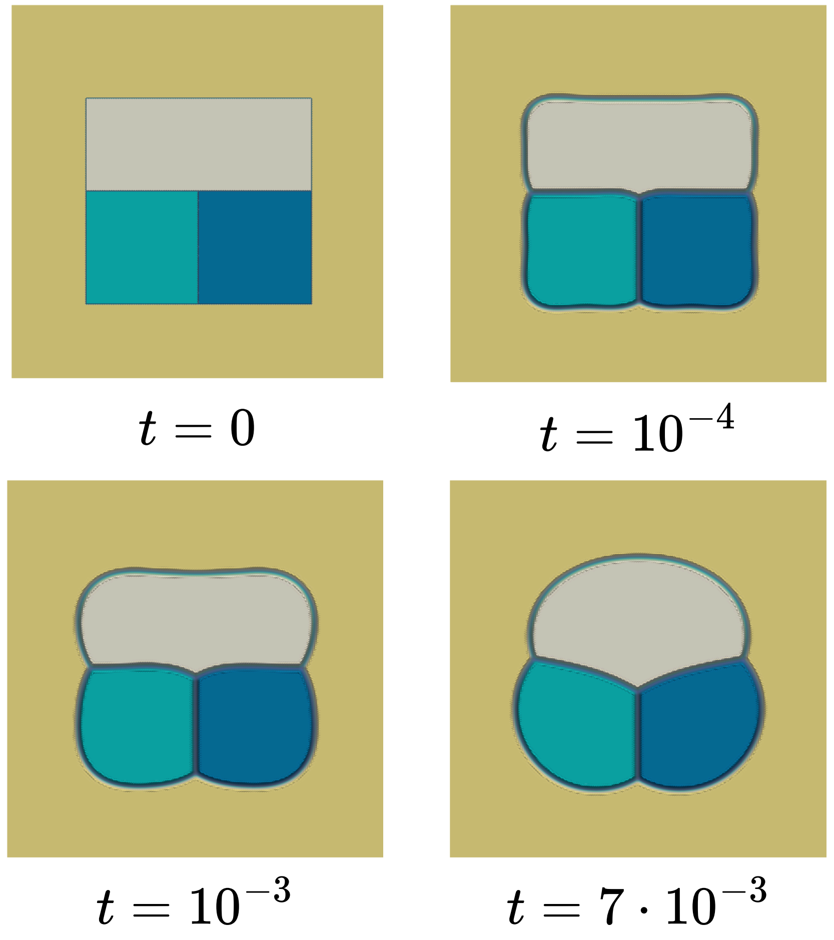
$$K = \{|\nabla u| \leq \phi\}$$

Elasto-Plastic Torsion & Eikonal Equation

Sum-To-One

$$K = \left\{u_i \geq 0, \sum u_i = 1\right\}$$

Multi-Phase/Material Problems



Four-Phase Cahn–Hilliard Flow

Other **Constraints**

Applications

- Obstacle Problem
- Contact Mechanics
- Variational Fracture
- Multi-Phase Flows
- Plasticity
- Eikonal Equation
- Monge–Ampère Equation
- Topology Optimization

...and more!

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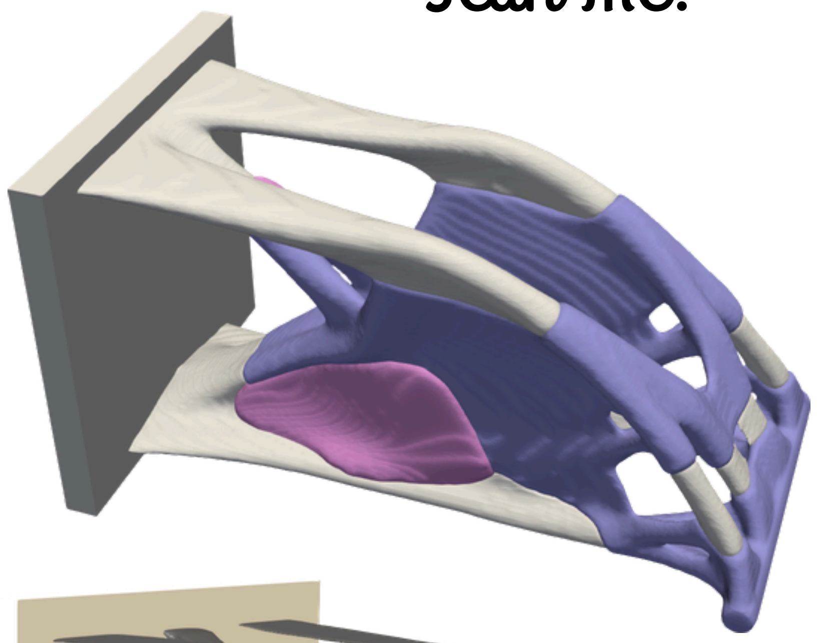
Gradient Constraints

$$K = \{|\nabla u| \leq \phi\}$$

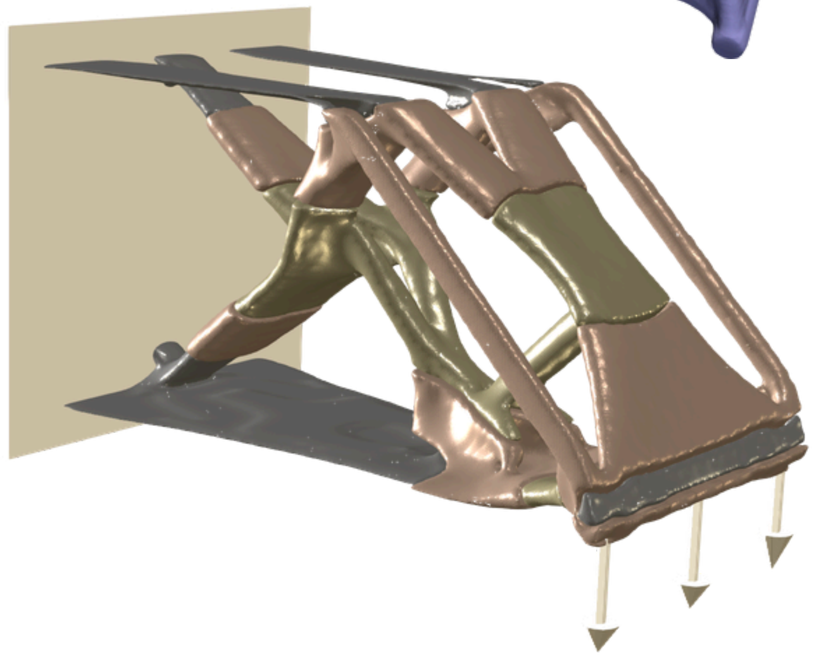
Elasto-Plastic Torsion & Eikonal Equation



Sum-To-One

$$K = \left\{u_i \geq 0, \sum u_i = 1\right\}$$

Multi-Phase/Material Problems



Multi-Material Topology Optimization

Other **Constraints**

Applications

- Obstacle Problem
- Contact Mechanics
- Variational Fracture
- Multi-Phase Flows
- Plasticity
- Eikonal Equation
- Monge–Ampère Equation
- Topology Optimization

...and more!

Scan Me!

Gradient Constraints

$$K = \{|\nabla u| \leq \phi\}$$

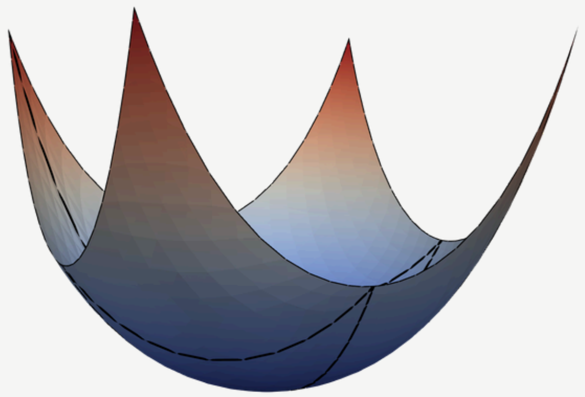
Eigenvalue (Unilateral)

$$K = \{\lambda_i(\nabla^2 u) \geq 0\}$$

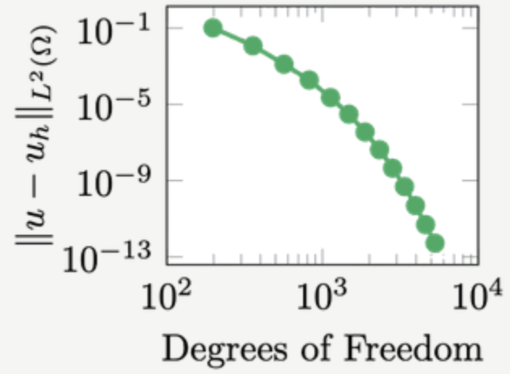
Monge–Ampère

Sum-To-One

$$K = \left\{u_i \geq 0, \sum u_i = 1\right\}$$



Computed solution to MA eqn.



$$\det(\nabla^2 u) = f \text{ in } \Omega$$
$$u = 0 \text{ on } \partial\Omega$$

Other **Constraints**

Applications

- Obstacle Problem
- Contact Mechanics
- Variational Fracture
- Multi-Phase Flows
- Plasticity
- Eikonal Equation
- Monge–Ampère Equation
- Topology Optimization

...and more!

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Gradient Constraints

$$K = \{|\nabla u| \leq \phi\}$$

Eigenvalue (Unilateral)

$$K = \{\lambda_i(\nabla^2 u) \geq 0\}$$

Monge–Ampère

Sum-To-One

$$K = \left\{u_i \geq 0, \sum u_i = 1\right\}$$

Eigenvalue (Bilateral)

$$K = \{-1 \leq \lambda_i(U) \leq 1\}$$

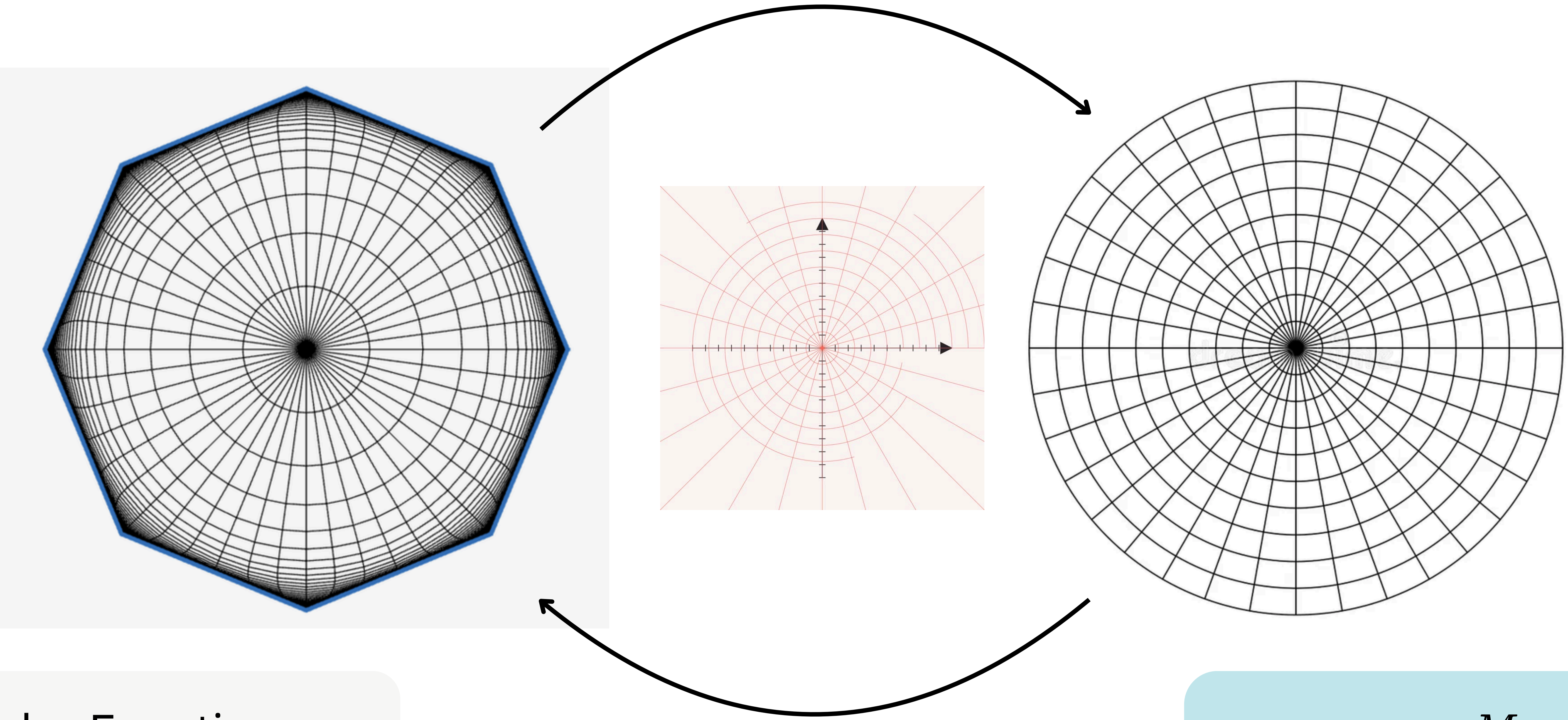
Liquid Crystals

Geometry Preserving **Transformations**

Vertices $\{v_i\}_{i=0}^{n} \subset \mathbb{R}^m$

$M = [v_0, v_1, \cdots, v_n] \in \mathbb{R}^{m \times n}$

∇R



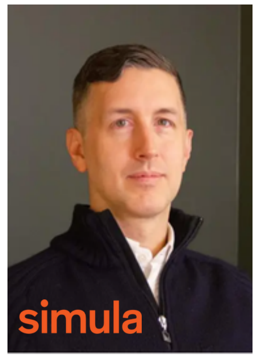
∇R^*

Legendre Functions
for Convex **Polytopes**

$$\nabla R^*(\psi) = \frac{M \exp(M^\top \psi)}{\sum_{j=1}^{n} (\exp(M^\top \psi))_j}$$

Multi-Angle **Topology Optimization**

Joint work with



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T.M. Surowiec

Sr. Research Scientist
P. Gangl

Postdoc
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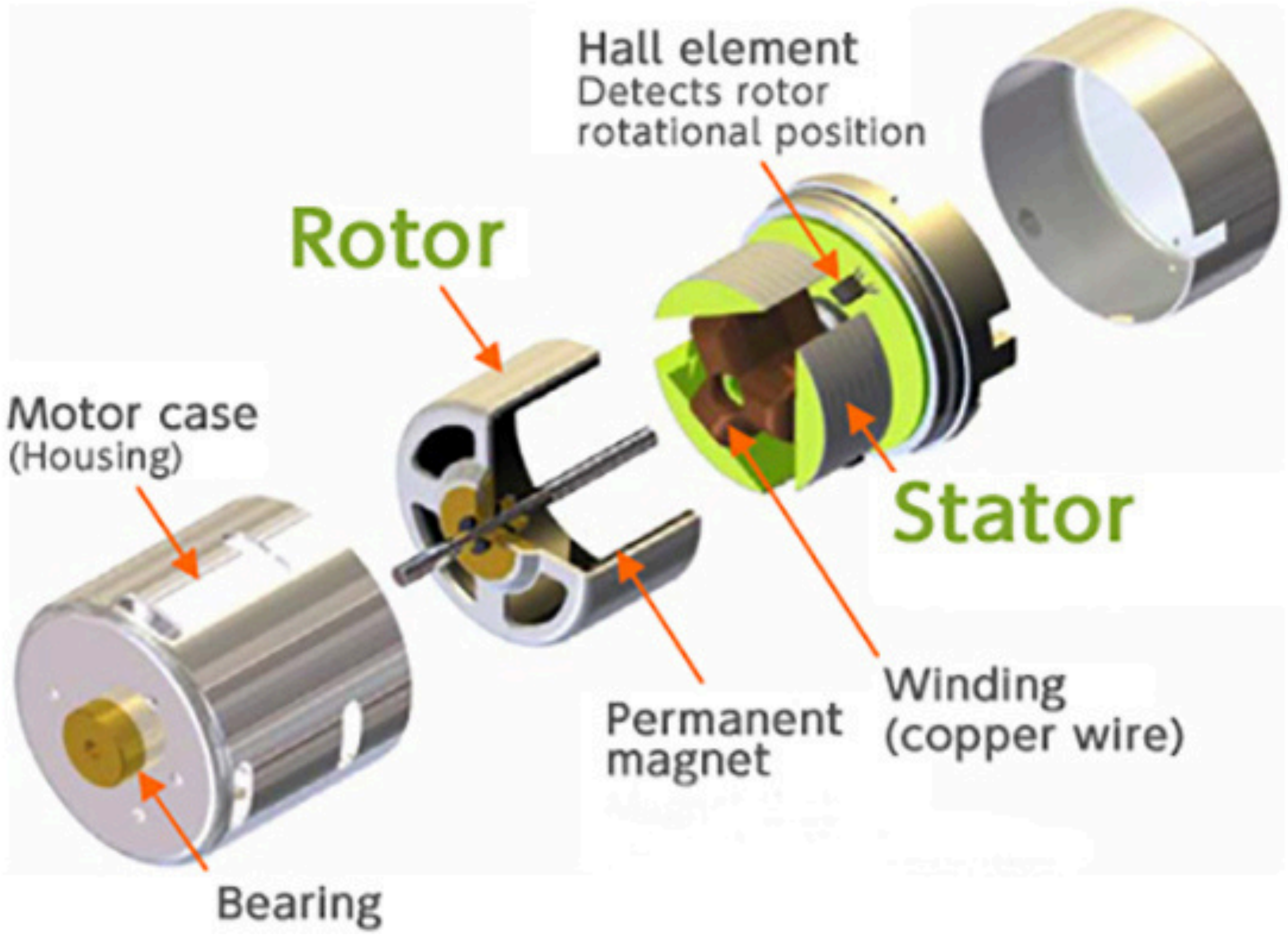
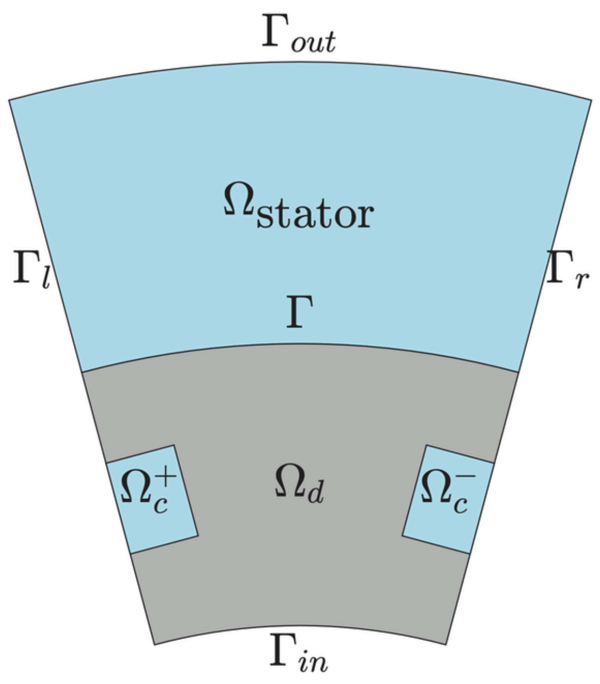
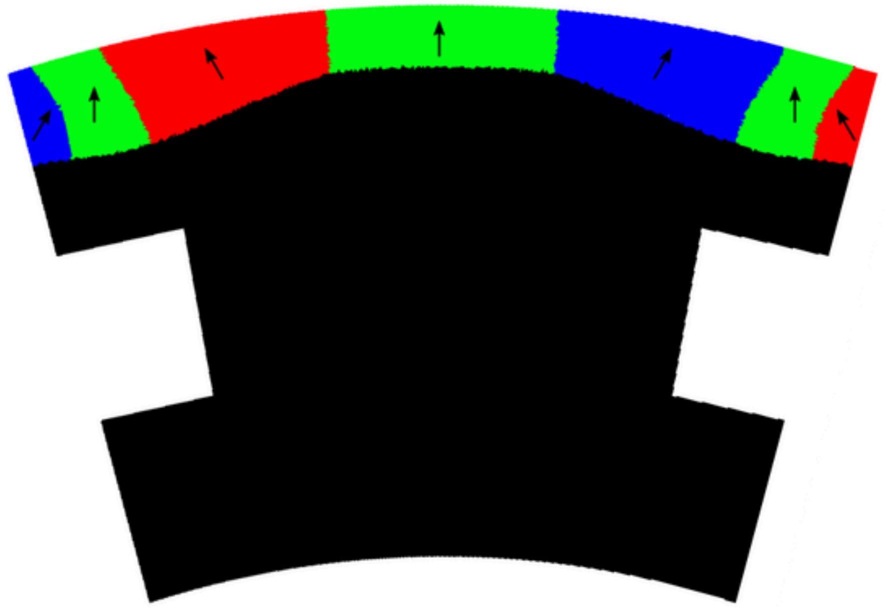
Research Engineer
B. Lazarov



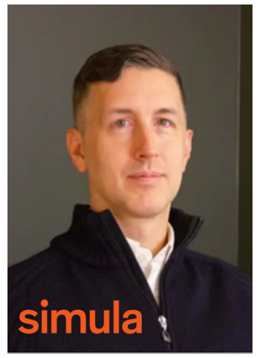
Electric Motor Design





Multi-Angle **Topology Optimization**

Joint work with



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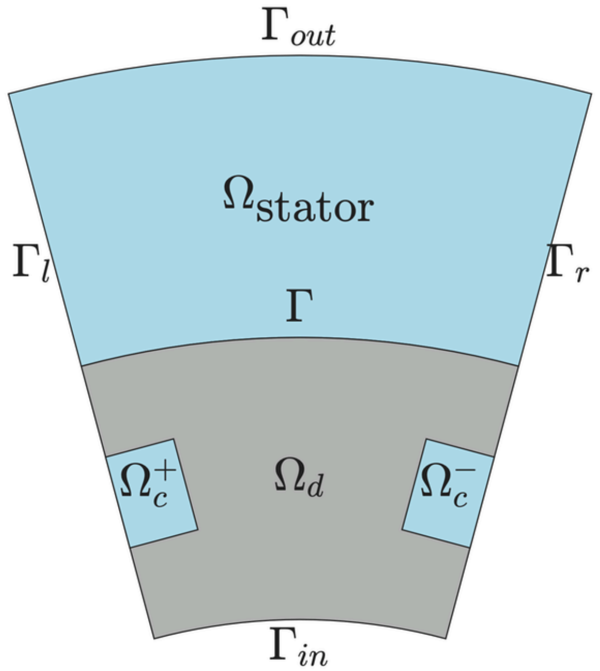
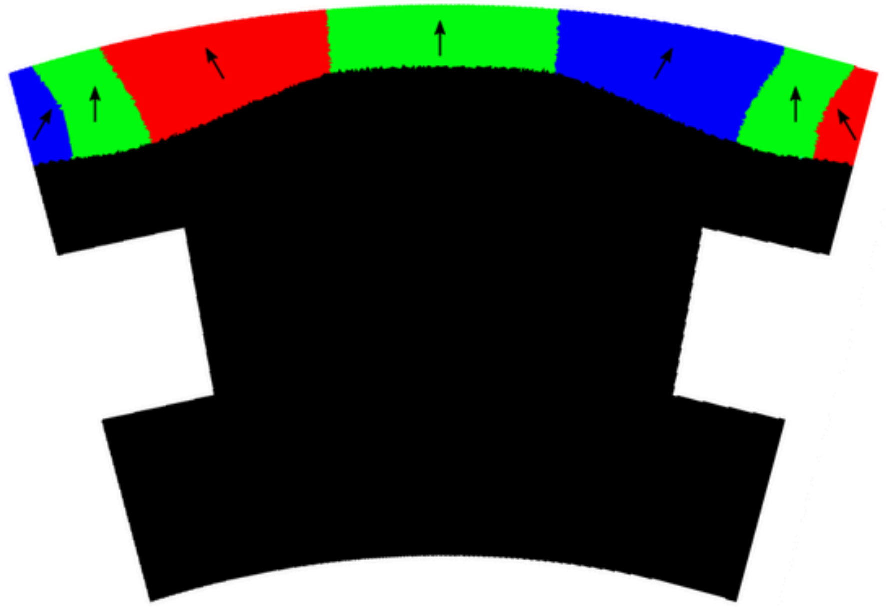
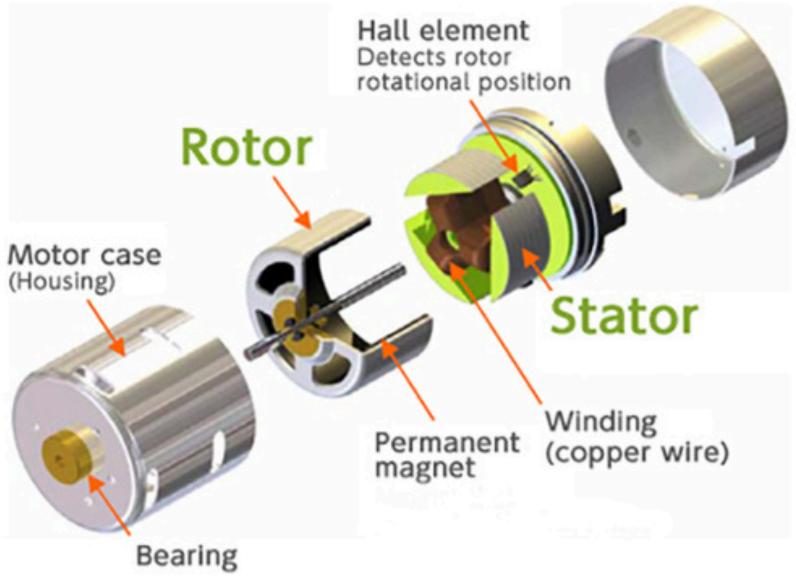
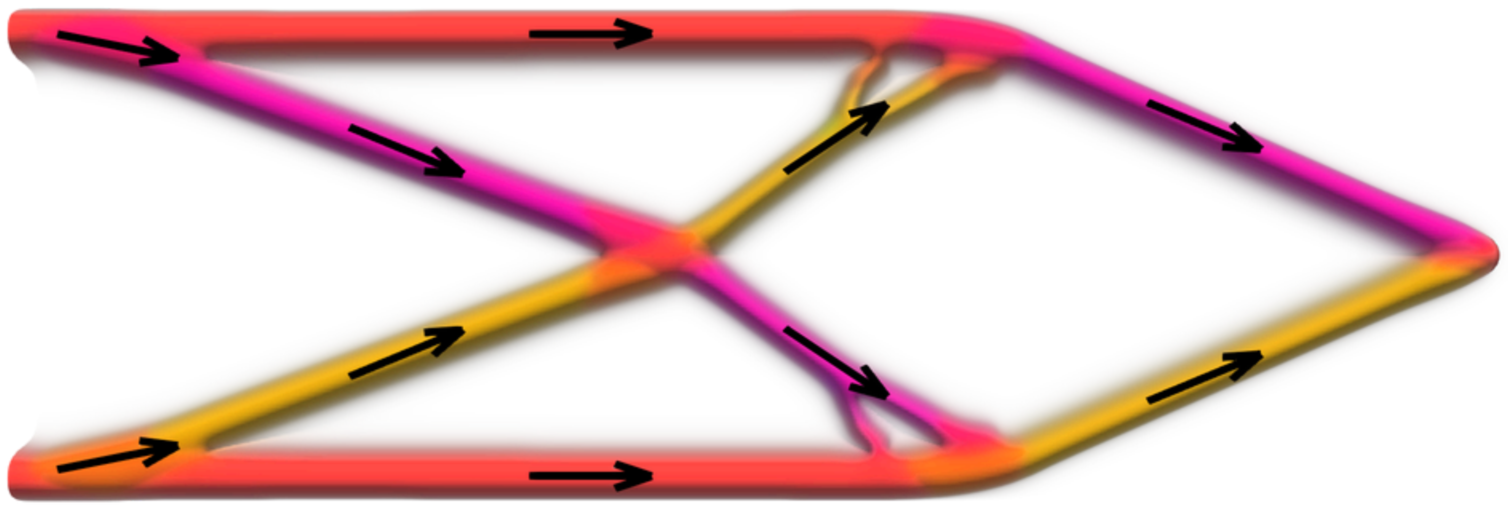
Electric Motor Design



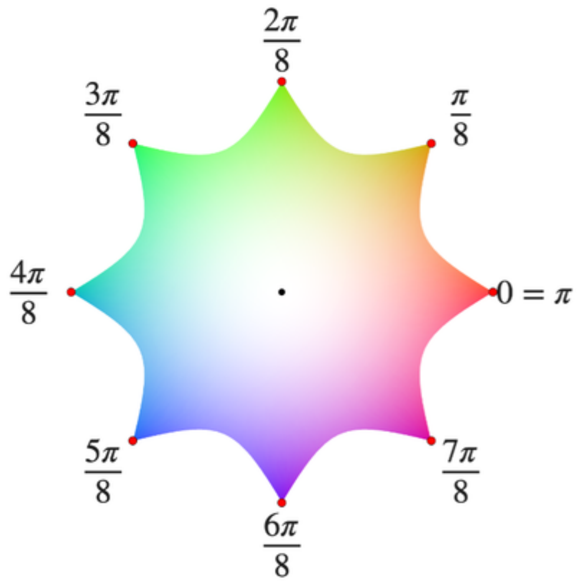




Structural Design



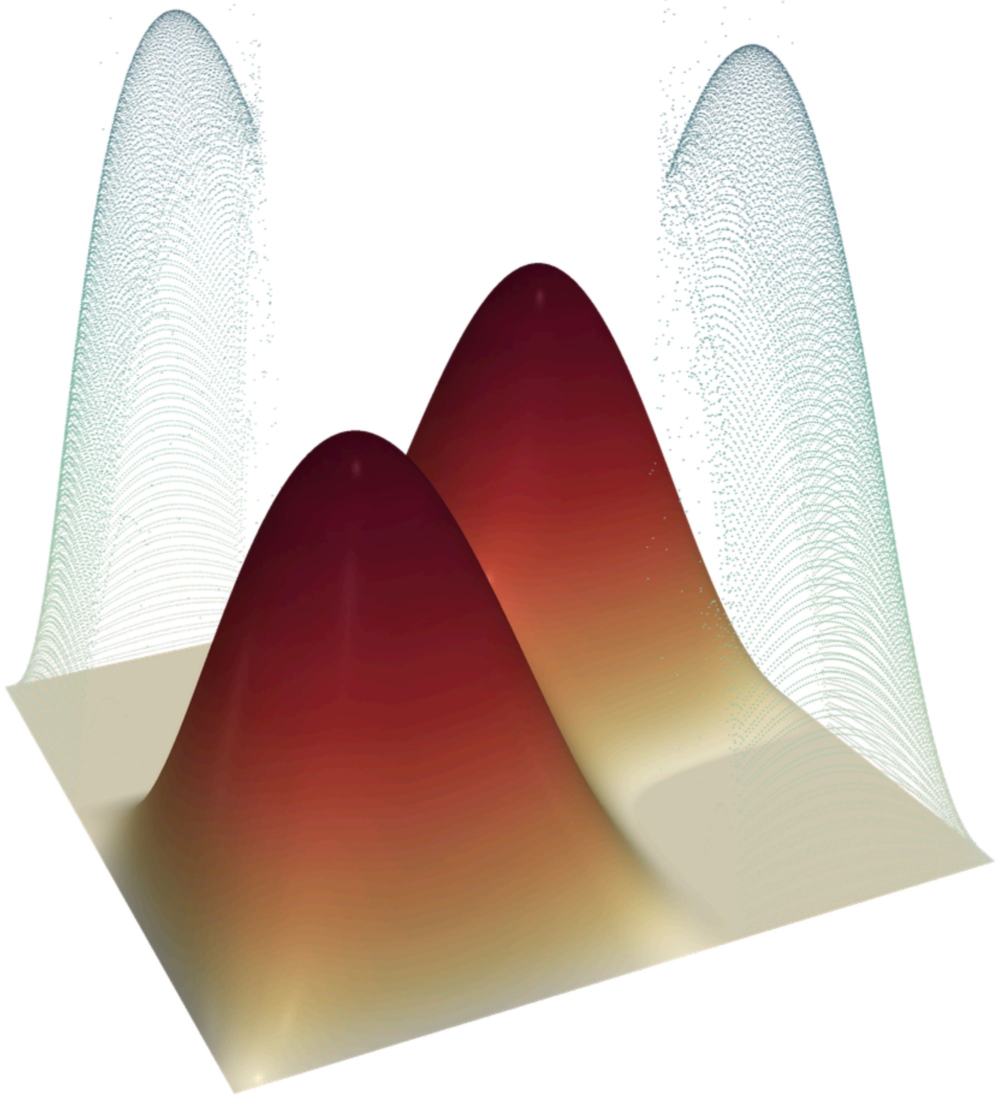
Optimal Stiffness Orientation



Obstacle Problem **Revisited**

$$\min_{u \in K} \ \frac{1}{2}\|\nabla u\|^2 - (f, u) \ \text{ where}$$

$$K = \{u \in H_0^1(\Omega) \mid u \geq 0 \text{ a.e. in } \Omega\}$$



Euler–Lagrange **Equations**

$$\alpha_k J'(u^k) + B^* \nabla R(Bu^k) - B^* \nabla R(Bu^{k-1}) = 0$$

Sequence of PDEs

$$-\Delta u^k + \alpha_k^{-1} \ln u^k = f + \alpha_k^{-1} \ln u^{k-1} \ \text{ in } \Omega$$

"Entropic Poisson equation"

$$u^k = 0 \ \text{ on } \partial\Omega$$

Contact **Revisited**

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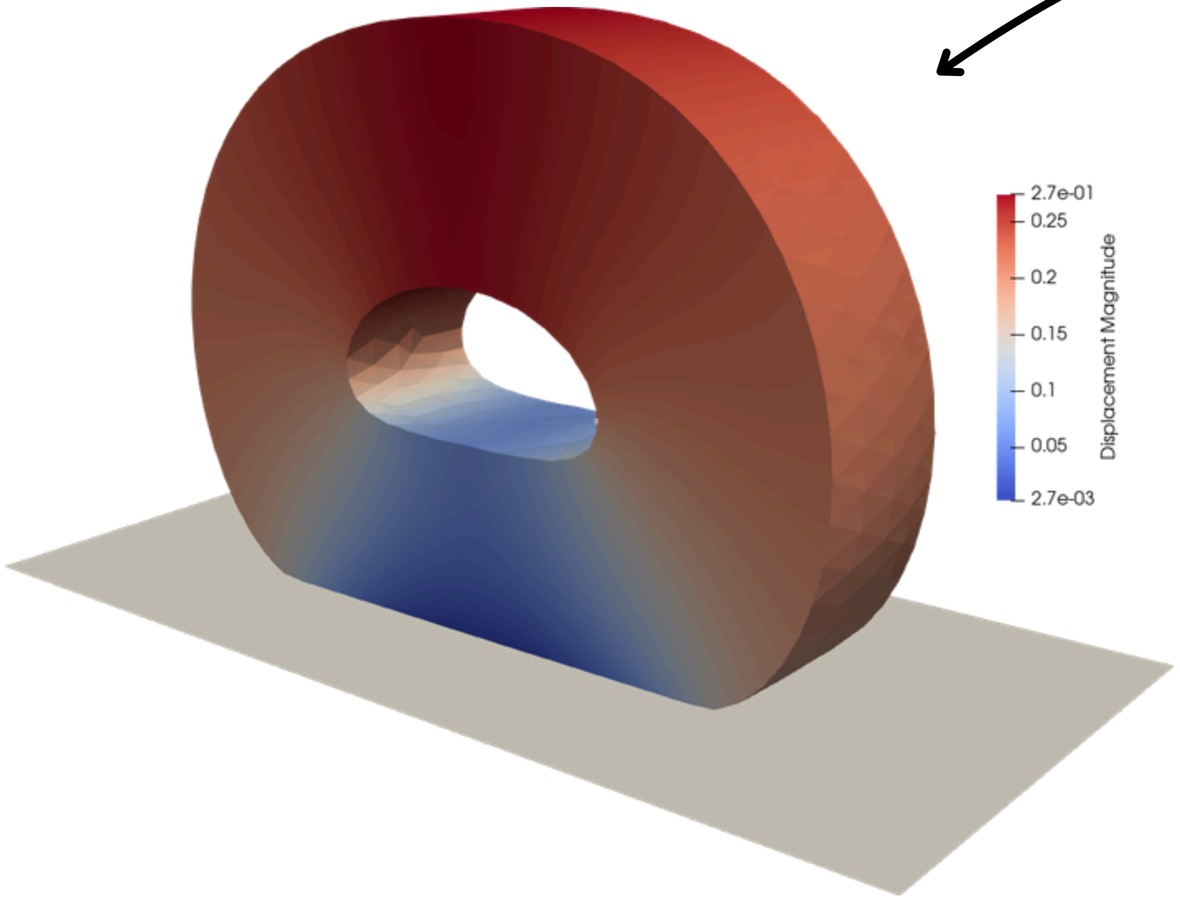
U.S. DEPARTMENT of ENERGY

Ph.D. Student
N. Reyes-Rivas

Professor
Y. Bazilevs



Sequence of PDEs

$$-\operatorname{Div}\sigma^k = f \ \text{ in } \Omega$$

$$u^k \cdot n - g = \exp\left(\sum_{i=0}^{k} \alpha_i p^i\right) \ \text{ on } \Gamma_{\mathrm{T}}$$

"Contact **Express**: The ***exp***onential ***press***ure method for contact mechanics"

Contact **Revisited**

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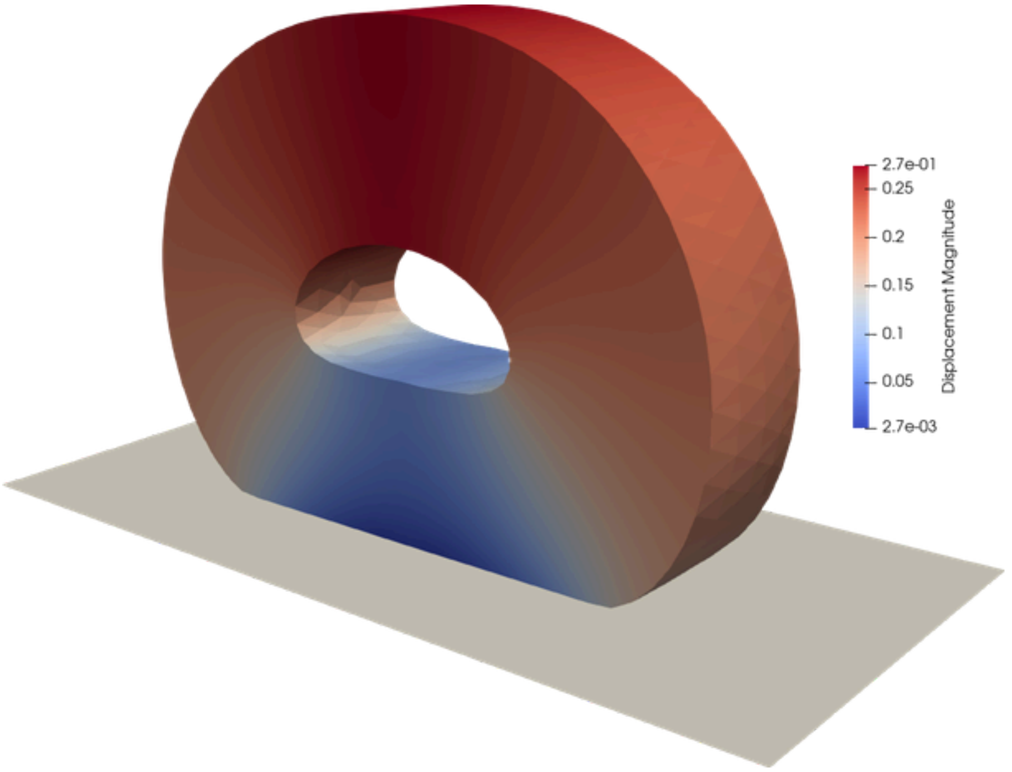
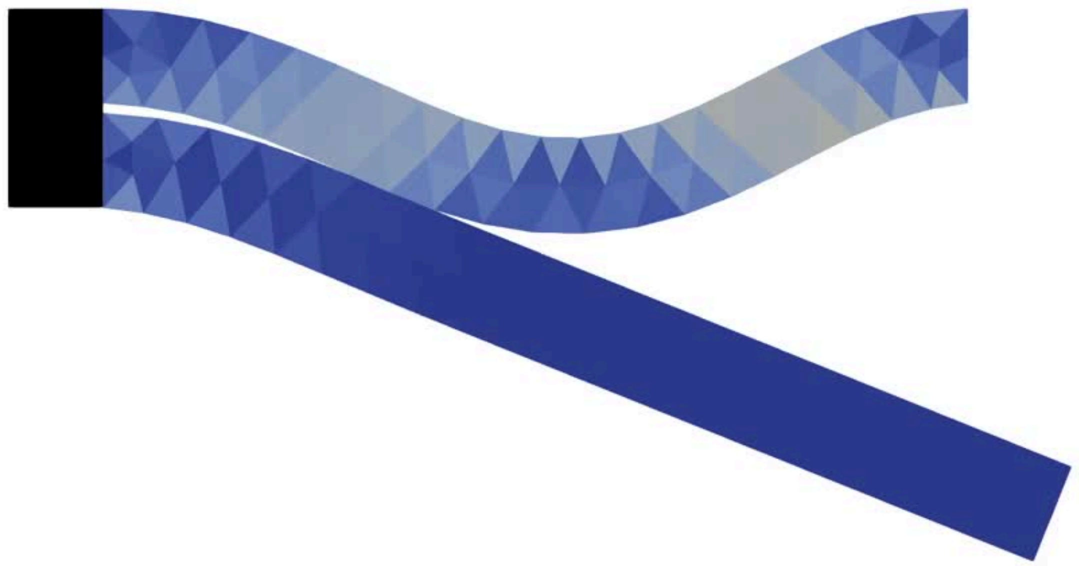
Contact **Express**



Sequence of PDEs

$$-\operatorname{Div}\sigma^k = f \ \text{ in } \Omega$$

$$u^k \cdot n - g = \exp\left(\sum_{i=0}^{k} \alpha_i p^i\right) \ \text{ on } \Gamma_T$$



Postdoc
I.P.A.
Papadopolous

Self/multi-body contact

$$K = K(u)$$

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Lawrence Livermore National Laboratory

Introduced

Unified Framework for Variational Problems with Inequality Constraints

Described

Theoretical Aspects and Convergence Guarantees

Presented

Numerous Applications, including

- Contact
- Fracture
- Topology Optimization
- Obstacle
- Multi-Material, etc.

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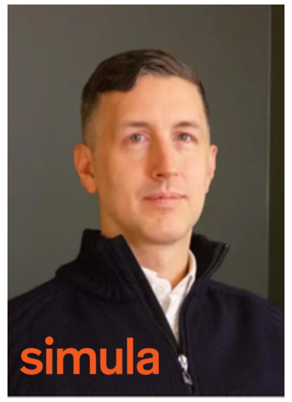

Chief Research Scientist
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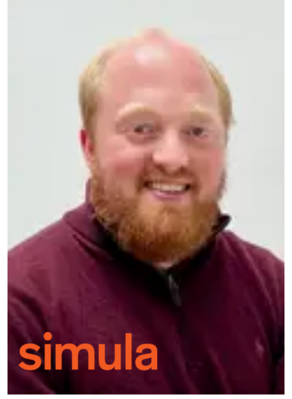

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Postdoc
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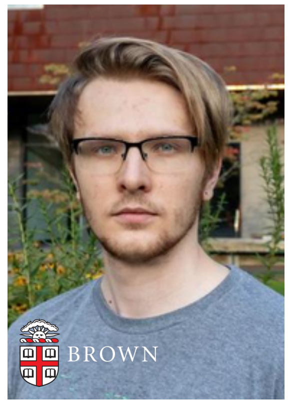


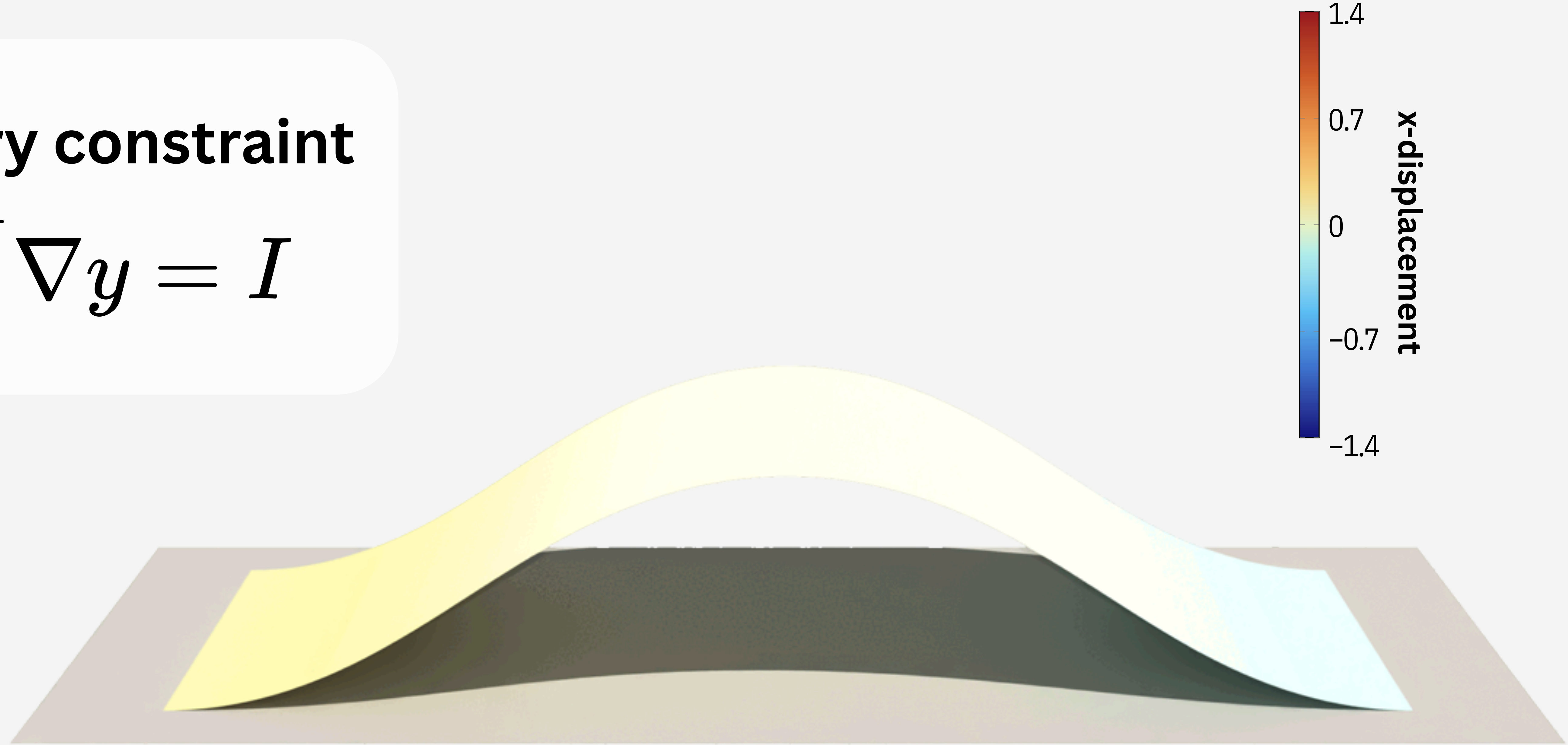
Ph.D. Student
A. Izmailov

You?

THANKS!

Isometry constraint

$$\nabla y^\top \nabla y = I$$



Next time: Manifold constraints; e.g., paper bending