Finite Element Exterior Calculus in Four-Dimensional Space with Applications

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Summary of the Talk

1. Overview of Finite Element Exterior Calculus
2. Finite Element Exterior Calculus in 4D
3. Applications in 4D
4. New Contributions in 4D
5. Numerical Experiments

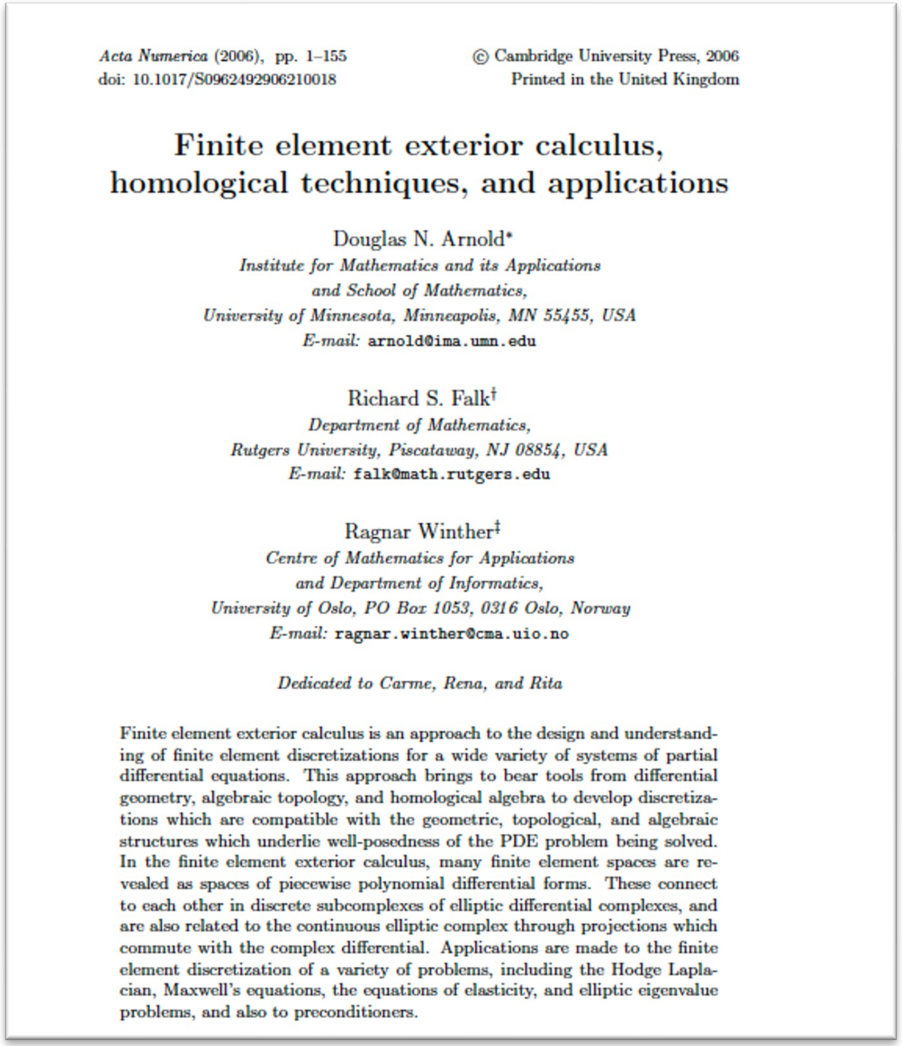
Overview of Finite Element Exterior Calculus

Finite Element Exterior Calculus (FEEC)

- **Question:** What is finite element exterior calculus?
- **Answer: "**A dimensionally-independent description of finite element methods which utilizes the language of differential geometry and algebraic topology."

- Developed by Arnold, Falk, and Winther, "Finite element exterior calculus, homological techniques, and applications" *Acta Numerica*, 2006
- Extended by Arnold, Falk, and Winther, "Finite element exterior calculus: from Hodge theory to numerical stability" *Bulletin of American Mathematical Society*, 2010
- Summarized by Arnold: "Finite Element Exterior Calculus" SIAM, 2018

Finite Element Exterior Calculus (FEEC)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Finite element exterior calculus, homological techniques, and applications

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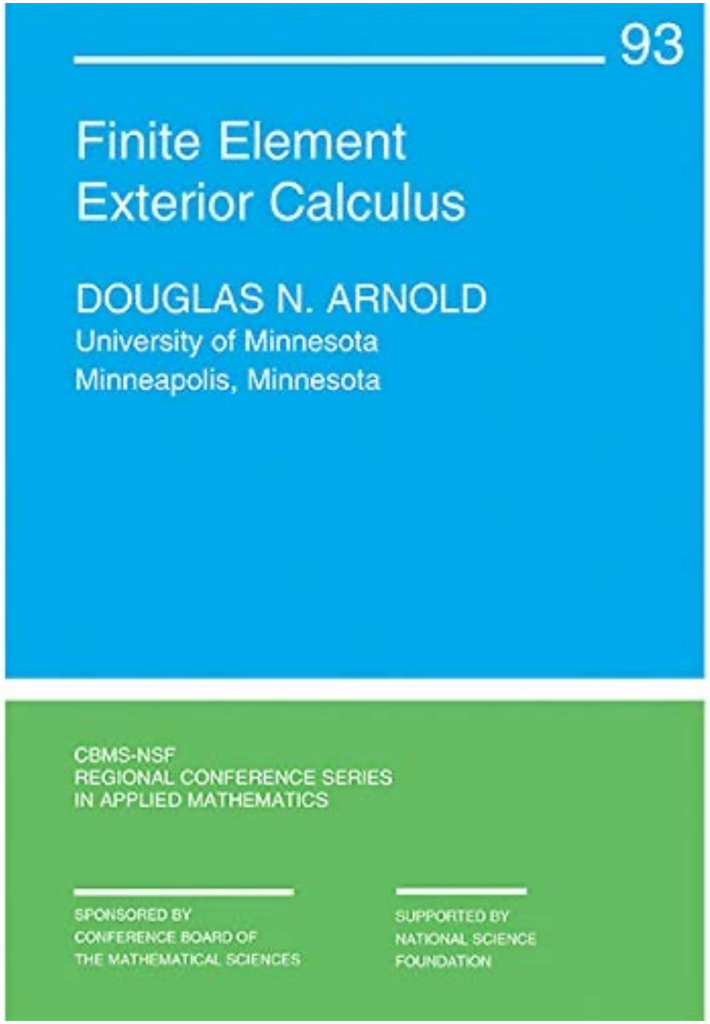
Dedicated to Carme, Rena, and Rita

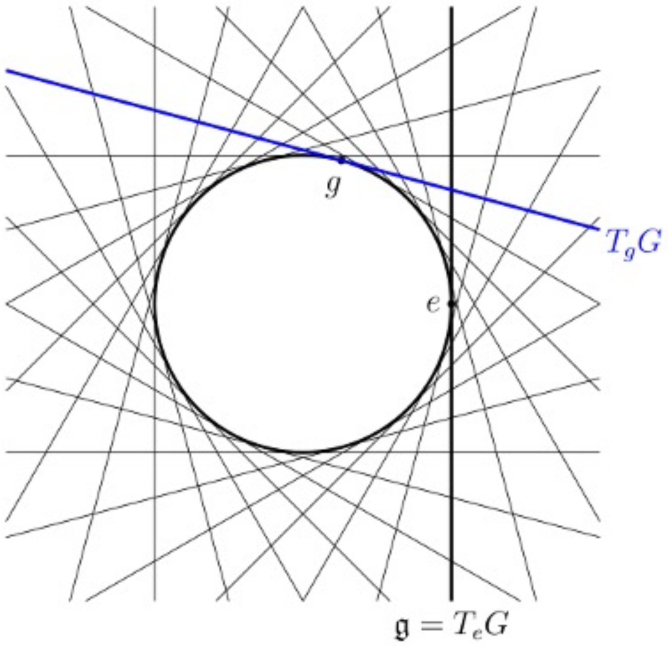
Finite element exterior calculus is an approach to the design and understanding of finite element discretizations for a wide variety of systems of partial differential equations. This approach brings to bear tools from differential geometry, algebraic topology, and homological algebra to develop discretizations which are compatible with the geometric, topological, and algebraic structures which underlie well-posedness of the PDE problem being solved. In the finite element exterior calculus, many finite element spaces are revealed as spaces of piecewise polynomial differential forms. These connect to each other in discrete subcomplexes of elliptic differential complexes, and are also related to the continuous elliptic complex through projections which commute with the complex differential. Applications are made to the finite element discretization of a variety of problems, including the Hodge Laplacian, Maxwell's equations, the equations of elasticity, and elliptic eigenvalue problems, and also to preconditioners.

155 pages



117 pages

Differential Forms



- Differential forms are the key building blocks of the theory

- **Question:** what is a differential form?
- Context:

V is a real vector space
$\mathrm{Alt}^s V$ is the space of alternating s-linear maps $V \times \cdots \times V \to \mathbb{R}$
Ω is a sufficiently smooth manifold

$T_x\Omega$ is the tangent space at each $x \in \Omega$
$T\Omega$ is the tangent bundle: all pairs (x, v) where $v \in T_x\Omega$
$\mathrm{Alt}^s T\Omega$ is the exterior forms bundle: all pairs (x, μ) where $\mu \in \mathrm{Alt}^s T_x\Omega$

Differential Forms

- Differential forms are the key building blocks of the theory

- **Question:** what is a differential form?
- Definition:

> A differential s-form ω is a map which associates $x \in \Omega$ with $\omega_x \in \mathrm{Alt}^s T_x \Omega$
> It is a subsection of the exterior forms bundle $\mathrm{Alt}^s T\Omega$

- In English: “It is a transformation or function which is a subset of the alternating *s*-linear maps acting on the tangent space.”

Differential Forms

- Smooth differential *s*-forms

$$\Lambda^s(\Omega) \text{ is the space of smooth differential forms}$$

- Precise definition:

$$\boxed{\begin{aligned} & v_1, \ldots, v_s \in T_x\Omega \\ & \omega_x(v_1, \ldots, v_s) \in \mathbb{R} \\ & \omega : x \mapsto \omega_x(v_1(x), \ldots, v_s(x)) \text{ is smooth} \end{aligned}}$$

Differential Forms

- Polynomial differential *s*-forms

$$P^k\Lambda^s(\Omega) \text{ is the space of polynomial differential } s\text{-forms}$$
$$\pi_k^s : \Lambda^s(\Omega) \to P^k\Lambda^s(\Omega)$$

- Polynomial differential *s*-form proxies

$$V_k\Lambda^s(\Omega) \text{ is the space of polynomial differential } s\text{-form proxies}$$
$$\Upsilon_s : P^k\Lambda^s(\Omega) \to V_k\Lambda^s(\Omega)$$

- A form proxy allows us to convert a differential form into a linear algebra object (scalar, vector, matrix, etc.)

Finite Element Exterior Calculus (FEEC)

- The power of FEEC: tensorial elements in *n*-dimensions
- Hypercube in *n*-dimensions is given by: $\mathfrak{H}^n$

$$P^k\Lambda^0(\mathfrak{H}^n) := \bigotimes_{j=1}^{n} P^k(\mathfrak{H}^1)$$

$$P^k\Lambda^s(\mathfrak{H}^n) := \bigoplus_{\sigma\in\Sigma(s,n)} \left[\bigotimes_{j=1}^{n} P^{k-\delta_{j,\sigma}}(\mathfrak{H}^1)\right] dx^{\sigma_1} \wedge dx^{\sigma_2} \wedge \cdots \wedge dx^{\sigma_s}$$

$$\delta_{j,\sigma} := \begin{cases} 1, & j \in \{\sigma_1, \sigma_2, \ldots, \sigma_s\}, \\ 0, & \text{otherwise.} \end{cases}$$

$\Sigma(s,n)$ set of increasing maps: $\{1,2,\ldots,s\} \to \{1,2,\ldots,n\} \quad s \leq n$

Finite Element Exterior Calculus (FEEC)

- The power of FEEC: tensorial elements in *n*-dimensions
- Degrees of freedom:

$$\left\{\omega \to \int_f \mathcal{R}[f](\omega) \wedge q, \qquad q \in \Lambda^{d-s}(f) \subset \mathcal{Q}^-_{k-1}(f), \qquad f \in \Delta_d(\mathfrak{H}^n),\, k \geq 1,\, 0 \leq s \leq n,\, d \geq s\right\},$$

$\Delta_d(\mathfrak{H}^n):$ d-dimensional faces of the n-dimensional hypercube

- **Question:** <u>How do we convince scientists and engineers to program this on a computer?</u>

Finite Element Exterior Calculus (FEEC)

- **Answer:** We transform the language of differential forms into the language of linear algebra using proxies!

- Example: *s*-forms in 3D

$$
\begin{array}{lll}
\text{0-forms,} & \omega \in \Lambda^0(\Omega) & \omega = \omega, \\
\text{1-forms,} & \omega \in \Lambda^1(\Omega) & \omega = \omega_1 dx^1 + \omega_2 dx^2 + \omega_3 dx^3, \\
\text{2-forms,} & \omega \in \Lambda^2(\Omega) & \omega = \omega_3 dx^1 \wedge dx^2 - \omega_2 dx^1 \wedge dx^3 + \omega_1 dx^2 \wedge dx^3, \\
\text{3-forms,} & \omega \in \Lambda^3(\Omega) & \omega = \omega_{123} dx^1 \wedge dx^2 \wedge dx^3
\end{array}
$$

Finite Element Exterior Calculus in 3D

- Example: *s*-form proxies in 3D

$$\Upsilon_0 \omega = \omega, \qquad \Upsilon_2 \omega = \begin{bmatrix} -\omega_1 \\ \omega_2 \\ -\omega_3 \end{bmatrix},$$

$$\Upsilon_1 \omega = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}, \quad \Upsilon_3 \omega = \omega_{123}$$

$$\widehat{\Upsilon}_2 \omega = \frac{1}{2} \begin{bmatrix} 0 & \omega_3 & \omega_2 \\ -\omega_3 & 0 & \omega_1 \\ -\omega_2 & -\omega_1 & 0 \end{bmatrix},$$

- These are just scalars and vectors
- **Question:** What can we do with this correspondence?

Finite Element Exterior Calculus in 3D

- Let's finally introduce some calculus!
- Exterior derivatives $d^{(m)}$ acting on *s*-forms in 3D

$$\boxed{\mathcal{D}'(\Omega,\Lambda^0) \xrightarrow{d^{(0)}} \mathcal{D}'(\Omega,\Lambda^1) \xrightarrow{d^{(1)}} \mathcal{D}'(\Omega,\Lambda^2) \xrightarrow{d^{(2)}} \mathcal{D}'(\Omega,\Lambda^3)}$$

- Derivative properties:

$$\boxed{\begin{array}{l} d^{(m)} \text{ exterior derivative of index } m \\ d^{(m)}\Lambda^m(\Omega) \subset \Lambda^{m+1}(\Omega) \\ d^{(m)}\left(d^{(m-1)}\cdot\right) = 0 \end{array}}$$

Finite Element Exterior Calculus in 3D

- Standard derivatives acting on *s*-form proxies in 3D

$$\boxed{\mathcal{D}'(\Omega, \mathbb{R}) \xrightarrow{\quad \nabla \quad} \mathcal{D}'(\Omega, \mathbb{R}^3) \xrightarrow{\quad \nabla\times \quad} \mathcal{D}'(\Omega, \mathbb{R}^3) \xrightarrow{\quad \nabla\cdot \quad} \mathcal{D}'(\Omega, \mathbb{R})}$$

- Gradient, curl, and divergence operators
- Derivative properties:

$$\boxed{\begin{aligned} &\nabla \times (\nabla \phi) = 0 \\ &\nabla \cdot \left(\nabla \times \vec{F}\right) = 0 \end{aligned}}$$

Finite Element Exterior Calculus in 3D

- First derivative operators on *s*-form proxies in 3D

gradient	$\nabla : \mathbb{R} \longrightarrow \mathbb{R}^3$
curl	$\nabla\times : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$
divergence	$\nabla\cdot : \mathbb{R}^3 \longrightarrow \mathbb{R}$

- Domains and ranges of these operators are always scalars or 3-vectors

Finite Element Exterior Calculus in 3D

- As promised, the proxy operators relate the two sequences

$$
\begin{array}{ccccccc}
\mathcal{D}'(\Omega, \Lambda^0) & \xrightarrow{d^{(0)}} & \mathcal{D}'(\Omega, \Lambda^1) & \xrightarrow{d^{(1)}} & \mathcal{D}'(\Omega, \Lambda^2) & \xrightarrow{d^{(2)}} & \mathcal{D}'(\Omega, \Lambda^3) \\
\Big\downarrow \Upsilon_0 & & \Big\downarrow \Upsilon_1 & & \Big\downarrow \Upsilon_2 & & \Big\downarrow \Upsilon_3 \\
\mathcal{D}'(\Omega, \mathbb{R}) & \xrightarrow{\nabla} & \mathcal{D}'(\Omega, \mathbb{R}^3) & \xrightarrow{\nabla \times} & \mathcal{D}'(\Omega, \mathbb{R}^3) & \xrightarrow{\nabla \cdot} & \mathcal{D}'(\Omega, \mathbb{R})
\end{array}
$$

- Let's rewrite the bottom sequence in more familiar notation

Finite Element Exterior Calculus in 3D

- de Rham complex in 3D

$$\boxed{C^\infty(\Omega,\mathbb{R}) \xrightarrow{\nabla} C^\infty(\Omega,\mathbb{R}^3) \xrightarrow{\nabla\times} C^\infty(\Omega,\mathbb{R}^3) \xrightarrow{\nabla\cdot} C^\infty(\Omega,\mathbb{R})}$$

- L^2 de Rham complex in 3D

$$\boxed{H(\nabla,\Omega,\mathbb{R}) \xrightarrow{\nabla} H(\nabla\times,\Omega,\mathbb{R}^3) \xrightarrow{\nabla\times} H(\nabla\cdot,\Omega,\mathbb{R}^3) \xrightarrow{\nabla\cdot} L^2(\Omega,\mathbb{R})}$$

$$\mathcal{D}'(\Omega,\mathbb{R}) \xrightarrow{\nabla} \mathcal{D}'(\Omega,\mathbb{R}^3) \xrightarrow{\nabla\times} \mathcal{D}'(\Omega,\mathbb{R}^3) \xrightarrow{\nabla\cdot} \mathcal{D}'(\Omega,\mathbb{R})$$

- **Question:** <u>What happens in four dimensions?</u>

Finite Element Exterior Calculus in 4D

Finite Element Exterior Calculus (FEEC)

- **Answer:** we introduce new differential forms, proxies, and derivative operators.

- Example: *s*-forms in 4D

$$
\begin{aligned}
&\text{0-forms,} && \omega \in \Lambda^0(\Omega), && \omega = \omega,\\
&\text{1-forms,} && \omega \in \Lambda^1(\Omega), && \omega = \omega_1 dx^1 + \omega_2 dx^2 + \omega_3 dx^3 + \omega_4 dx^4,\\
&\text{2-forms,} && \omega \in \Lambda^2(\Omega), && \omega = \omega_{12} dx^1 \wedge dx^2 + \omega_{13} dx^1 \wedge dx^3 + \omega_{14} dx^1 \wedge dx^4\\
& && && \quad + \omega_{23} dx^2 \wedge dx^3 + \omega_{24} dx^2 \wedge dx^4 + \omega_{34} dx^3 \wedge dx^4,\\
&\text{3-forms,} && \omega \in \Lambda^3(\Omega), && \omega = \omega_{123} dx^1 \wedge dx^2 \wedge dx^3 + \omega_{124} dx^1 \wedge dx^2 \wedge dx^4\\
& && && \quad + \omega_{134} dx^1 \wedge dx^3 \wedge dx^4 + \omega_{234} dx^2 \wedge dx^3 \wedge dx^4,\\
&\text{4-forms,} && \omega \in \Lambda^4(\Omega), && \omega = \omega_{1234} dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4.
\end{aligned}
$$

Finite Element Exterior Calculus in 4D

- Example: *s*-form proxies in 4D

$$
\begin{aligned}
&\Upsilon_0\omega = \omega,\\
&\Upsilon_1\omega = \begin{bmatrix}\omega_1\\ \omega_2\\ \omega_3\\ \omega_4\end{bmatrix}, \quad \Upsilon_2\omega = \frac{1}{2}\begin{bmatrix}0 & \omega_{12} & \omega_{13} & \omega_{14}\\ -\omega_{12} & 0 & \omega_{23} & \omega_{24}\\ -\omega_{13} & -\omega_{23} & 0 & \omega_{34}\\ -\omega_{14} & -\omega_{24} & -\omega_{34} & 0\end{bmatrix}, \quad \Upsilon_3\omega = \begin{bmatrix}\omega_{234}\\ -\omega_{134}\\ \omega_{124}\\ -\omega_{123}\end{bmatrix},\\
&\Upsilon_4\omega = \omega_{1234}.
\end{aligned}
$$

- These are scalars, vectors, and a skew-symmetric matrix
- **Question:** What can we do with this correspondence?

Finite Element Exterior Calculus in 4D

- Exterior derivatives $d^{(m)}$ acting on s-forms in 4D

$$\mathcal{D}'(\Omega, \Lambda^0) \xrightarrow{d^{(0)}} \mathcal{D}'(\Omega, \Lambda^1) \xrightarrow{d^{(1)}} \mathcal{D}'(\Omega, \Lambda^2) \xrightarrow{d^{(2)}} \mathcal{D}'(\Omega, \Lambda^3) \xrightarrow{d^{(3)}} \mathcal{D}'(\Omega, \Lambda^4)$$

- Standard derivatives acting on s-form proxies in 4D

$$\mathcal{D}'(\Omega, \mathbb{R}) \xrightarrow{\text{grad}} \mathcal{D}'(\Omega, \mathbb{R}^4) \xrightarrow{\text{skwGrad}} \mathcal{D}'(\Omega, \mathbb{K}) \xrightarrow{\text{curl}} \mathcal{D}'(\Omega, \mathbb{R}^4) \xrightarrow{\text{div}} \mathcal{D}'(\Omega, \mathbb{R})$$

- grad, skew-gradient, curl, and divergence operators
- $\mathbb{K}$ is the field of skew-symmetric matrices

Finite Element Exterior Calculus in 4D

- First derivative operators on *s*-form proxies in 4D

gradient	$\mathrm{grad} : \mathbb{R} \longrightarrow \mathbb{R}^4$
skew-gradient	$\mathrm{skwGrad} : \mathbb{R}^4 \longrightarrow \mathbb{K}$
curl	$\mathrm{curl} : \mathbb{K} \longrightarrow \mathbb{R}^4$
divergence	$\mathrm{div} : \mathbb{R}^4 \longrightarrow \mathbb{R}$

- Domains and ranges of these operators are always scalars, 4-vectors, or 4x4 skew-symmetric matrices

Finite Element Exterior Calculus in 4D

- New derivative operators $u \in L^2(\Omega, \mathbb{R}),\ E \in L^2(\Omega, \mathbb{R}^4)$

$d^{(0)}$

$$[\operatorname{grad} u]_i = \partial_i u,$$
$$[\operatorname{Grad} E]_{ij} = \partial_j E_i \qquad i = 1, \ldots, 4, \quad j = 1, \ldots, 4$$

$d^{(1)}$

$$[\operatorname{skwGrad} E] = \frac{1}{2}\Big([\operatorname{Grad} E]^T - [\operatorname{Grad} E]\Big)$$

Finite Element Exterior Calculus in 4D

- New derivative operators $F \in L^2(\Omega, \mathbb{K}),\ G \in L^2(\Omega, \mathbb{R}^4)$

$$d^{(2)} \qquad [\operatorname{curl} F]_i = \sum_{k,l=1}^{4} \varepsilon_{ijkl} \partial_j F_{kl}$$

$$d^{(3)} \qquad [\operatorname{div} G] = \partial_i G_i$$

Finite Element Exterior Calculus in 4D

- The proxy operators relate the two sequences

$$
\begin{tikzcd}
\mathcal{D}'(\Omega, \Lambda^0) \arrow[r, "d^{(0)}"] \arrow[d, "\Upsilon_0"] & \mathcal{D}'(\Omega, \Lambda^1) \arrow[r, "d^{(1)}"] \arrow[d, "\Upsilon_1"] & \mathcal{D}'(\Omega, \Lambda^2) \arrow[r, "d^{(2)}"] \arrow[d, "\Upsilon_2"] & \mathcal{D}'(\Omega, \Lambda^3) \arrow[r, "d^{(3)}"] \arrow[d, "\Upsilon_3"] & \mathcal{D}'(\Omega, \Lambda^4) \arrow[d, "\Upsilon_4"] \\
\mathcal{D}'(\Omega, \mathbb{R}) \arrow[r, "\mathrm{grad}"] & \mathcal{D}'(\Omega, \mathbb{R}^4) \arrow[r, "\mathrm{skwGrad}"] & \mathcal{D}'(\Omega, \mathbb{K}) \arrow[r, "\mathrm{curl}"] & \mathcal{D}'(\Omega, \mathbb{R}^4) \arrow[r, "\mathrm{div}"] & \mathcal{D}'(\Omega, \mathbb{R})
\end{tikzcd}
$$

- Let’s rewrite the bottom sequence in more familiar notation

Finite Element Exterior Calculus in 4D

- de Rham complex in 4D

$$\boxed{C^\infty(\Omega,\mathbb{R}) \xrightarrow{\text{grad}} C^\infty(\Omega,\mathbb{R}^4) \xrightarrow{\text{skwGrad}} C^\infty(\Omega,\mathbb{K}) \xrightarrow{\text{curl}} C^\infty(\Omega,\mathbb{R}^4) \xrightarrow{\text{div}} C^\infty(\Omega,\mathbb{R})}$$

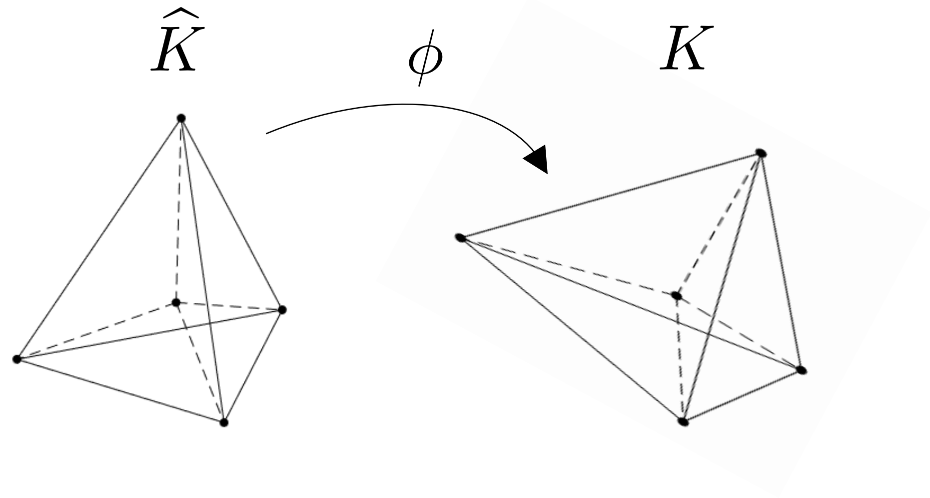
- *L^2* de Rham complex in 4D

$$\boxed{H(\text{grad},\Omega,\mathbb{R}) \xrightarrow{\text{grad}} H(\text{skwGrad},\Omega,\mathbb{R}^4) \xrightarrow{\text{skwGrad}} H(\text{curl},\Omega,\mathbb{K}) \xrightarrow{\text{curl}} H(\text{div},\Omega,\mathbb{R}^4) \xrightarrow{\text{div}} L^2(\Omega,\mathbb{R}).}$$

$$\mathcal{D}'(\Omega,\mathbb{R}) \xrightarrow{\text{grad}} \mathcal{D}'(\Omega,\mathbb{R}^4) \xrightarrow{\text{skwGrad}} \mathcal{D}'(\Omega,\mathbb{K}) \xrightarrow{\text{curl}} \mathcal{D}'(\Omega,\mathbb{R}^4) \xrightarrow{\text{div}} \mathcal{D}'(\Omega,\mathbb{R})$$

- **Question:** <u>What happens in five dimensions?</u>

Finite Element Exterior Calculus in 4D



- Pullback operations (Piola transformations)
- Consider a mapping function $\phi : \widehat{K} \to K$
- Jacobian: $[D\phi]_{ij} = \partial_j \phi_i$

- Then:

$$
\begin{aligned}
u &= \Upsilon_0 \omega, \quad \forall u \in H\left(\mathrm{grad}, \Omega, \mathbb{R}\right), && \Upsilon_0 \phi^* \omega = u \circ \phi, \\
E &= \Upsilon_1 \omega, \quad \forall E \in H\left(\mathrm{skwGrad}, \Omega, \mathbb{R}^4\right), && \Upsilon_1 \phi^* \omega = D\phi^T \left[E \circ \phi\right], \\
F &= \Upsilon_2 \omega, \quad \forall F \in H\left(\mathrm{curl}, \Omega, \mathbb{K}\right), && \Upsilon_2 \phi^* \omega = D\phi^T \left[F \circ \phi\right] D\phi, \\
G &= \Upsilon_3 \omega, \quad \forall G \in H\left(\mathrm{div}, \Omega, \mathbb{R}^4\right), && \Upsilon_3 \phi^* \omega = \left|D\phi\right| D\phi^{-1} \left[G \circ \phi\right], \\
q &= \Upsilon_4 \omega, \quad \forall q \in L^2\left(\Omega, \mathbb{R}\right), && \Upsilon_4 \phi^* \omega = \left|D\phi\right| \left[q \circ \phi\right]
\end{aligned}
$$

Summary: 3D vs. 4D

- Key idea: 3D is not 4D

- curl operator is fundamentally different
  - curl in 3D: maps 3-vectors to 3-vectors
  - curl in 4D: maps 4x4 skew-symmetric matrices to 4-vectors

- skew-gradient operator is new and unique
  - skwGrad in 3D: maps 3-vectors to 3x3 skew-symmetric matrices
  - skwGrad in 4D: maps 4-vectors to 4x4 skew-symmetric matrices

Summary: 3D vs. 4D

- skwGrad in 3D contains the same information as the curl!

$$\boxed{\begin{aligned} [\mathrm{Grad}\,\mathcal{E}]_{ij} &= \partial_j \mathcal{E}_i \qquad i = 1, \ldots, 3, \quad j = 1, \ldots, 3 \\ [\mathrm{skwGrad}\,\mathcal{E}] &= \frac{1}{2}\Big([\mathrm{Grad}\,\mathcal{E}]^T - [\mathrm{Grad}\,\mathcal{E}]\Big) \end{aligned}}$$

$$\boxed{\begin{aligned} &\nabla \times \mathcal{E} = 2w \\ &w_1 = [\mathrm{skwGrad}\,\mathcal{E}]_{23}\,, \quad w_2 = [\mathrm{skwGrad}\,\mathcal{E}]_{31}\,, \quad w_3 = [\mathrm{skwGrad}\,\mathcal{E}]_{12} \end{aligned}}$$

- Not true in 4D: cannot identify 4x4 skew-symmetric matrix with a 4-vector

Applications in 4D

Applications in 4D

- **Question:** But why is 4D useful?
- **Answer:** the inhomogeneous Maxwell's equations

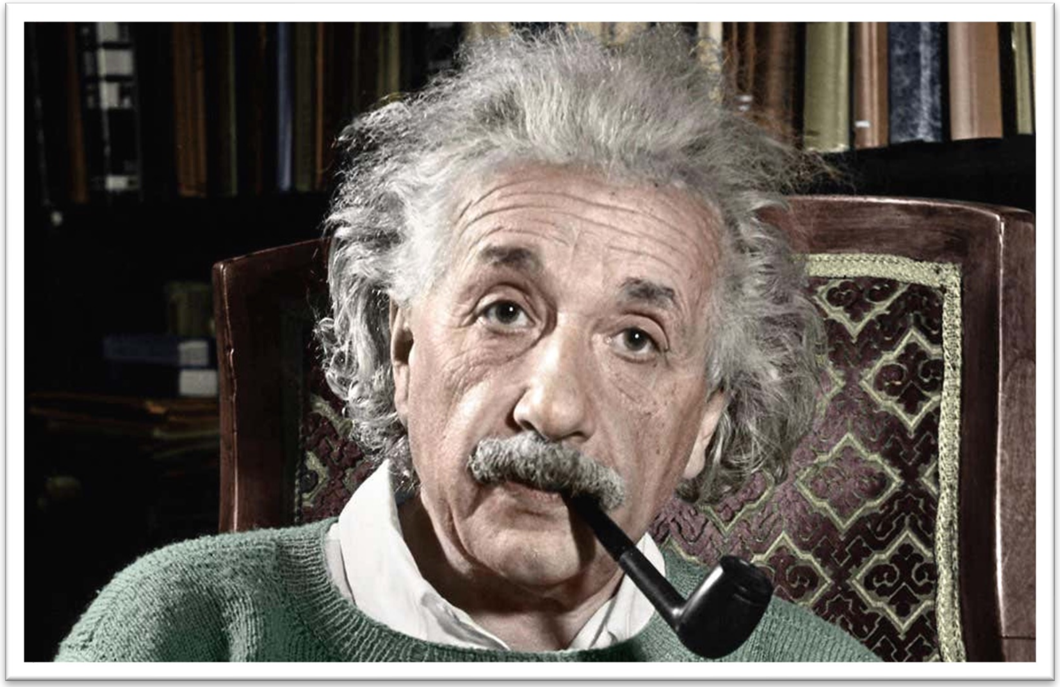
$$\boxed{\begin{aligned} \nabla \cdot E &= 4\pi\rho, \qquad & \nabla \cdot B &= 0, \\ c\nabla \times E + \frac{\partial}{\partial t}B &= 0, \qquad & c\nabla \times B - \frac{\partial}{\partial t}E &= 4\pi J \end{aligned}}$$

- Electric and magnetic fields $E = (E_x, E_y, E_z), \quad B = (B_x, B_y, B_z)$
- Electric current density $J = (j_x, j_y, j_z)$

Applications in 4D

- **Problem:**
  - Einstein: none of the 1-forms or 2-forms for the electric and magnetic fields in 3D have an observer independent existence
  - Observers in relative motion disagree about these forms, and also about space and time

- **Solution:** Construct 2-forms and 3-forms for the electric and magnetic fields in 4D



Applications in 4D

- 2-form for the electric and magnetic fields (Maxwell form)

$$\boxed{\begin{aligned}\omega = -c(B_x dx^1 \wedge dx^2 + B_y dx^1 \wedge dx^3 + B_z dx^1 \wedge dx^4) \\ - E_x dx^3 \wedge dx^4 + E_y dx^2 \wedge dx^4 - E_z dx^2 \wedge dx^3\end{aligned}}$$

- 2-form for the electric and magnetic fields (Faraday form)

$$\boxed{\begin{aligned}\varphi = B_x dx^3 \wedge dx^4 - B_y dx^2 \wedge dx^4 + B_z dx^2 \wedge dx^3 \\ - c(E_x dx^1 \wedge dx^2 + E_y dx^1 \wedge dx^3 + E_z dx^1 \wedge dx^4)\end{aligned}}$$

Applications in 4D

- 3-form for the electric current density

$$\boxed{\begin{aligned}\sigma = -\rho dx^2 \wedge dx^3 \wedge dx^4 + j_x dx^1 \wedge dx^3 \wedge dx^4 \\ - j_y dx^1 \wedge dx^2 \wedge dx^4 + j_z dx^1 \wedge dx^2 \wedge dx^3\end{aligned}}$$

Applications in 4D

- Form proxies $F = \Upsilon_2 \omega, \quad G = \Upsilon_3 \sigma, \quad H = \Upsilon_2 \varphi$

$$F = \frac{1}{2} \begin{bmatrix} 0 & -cB_x & -cB_y & -cB_z \\ cB_x & 0 & -E_z & E_y \\ cB_y & E_z & 0 & -E_x \\ cB_z & -E_y & E_x & 0 \end{bmatrix}, \quad H = \frac{1}{2} \begin{bmatrix} 0 & -cE_x & -cE_y & -cE_z \\ cE_x & 0 & B_z & -B_y \\ cE_y & -B_z & 0 & B_x \\ cE_z & B_y & -B_x & 0 \end{bmatrix}, \quad G = - \begin{bmatrix} \rho \\ j_x \\ j_y \\ j_z \end{bmatrix}$$

- Final equations

$$\boxed{\operatorname{curl}(F) = 4\pi G, \qquad \operatorname{curl}(H) = 0, \qquad \operatorname{div}(G) = 0}$$

Applications in 4D

$$\mathrm{curl}(F) = 4\pi G$$

$$\begin{bmatrix} \nabla \cdot E \\ c\nabla \times B - \frac{\partial E}{\partial t} \end{bmatrix} = 4\pi \begin{bmatrix} \rho \\ J \end{bmatrix}$$

$$\mathrm{curl}(H) = 0$$

$$\begin{bmatrix} \nabla \cdot B \\ c\nabla \times E + \frac{\partial B}{\partial t} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\mathrm{div}(G) = 0$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot J = 0$$

Applications in 4D

- **Utility:** We need to construct finite-dimensional subspaces of the infinite-dimensional Sobolev spaces for the curl and divergence in 4D

$$
\begin{aligned}
H\left(\mathrm{grad}, \Omega, \mathbb{R}\right) &= \left\{u \in L^2\left(\Omega, \mathbb{R}\right) : \mathrm{grad}\, u \in L^2\left(\Omega, \mathbb{R}^4\right)\right\}, \\
H\left(\mathrm{skwGrad}, \Omega, \mathbb{R}^4\right) &= \left\{E \in L^2\left(\Omega, \mathbb{R}^4\right) : \mathrm{skwGrad}\, E \in L^2\left(\Omega, \mathbb{K}\right)\right\}, \\
H\left(\mathrm{curl}, \Omega, \mathbb{K}\right) &= \left\{F \in L^2\left(\Omega, \mathbb{K}\right) : \mathrm{curl}\, F \in L^2\left(\Omega, \mathbb{R}^4\right)\right\}, \\
H\left(\mathrm{div}, \Omega, \mathbb{R}^4\right) &= \left\{G \in L^2\left(\Omega, \mathbb{R}^4\right) : \mathrm{div}\, G \in L^2\left(\Omega, \mathbb{R}\right)\right\}
\end{aligned}
$$

New Contributions in 4D

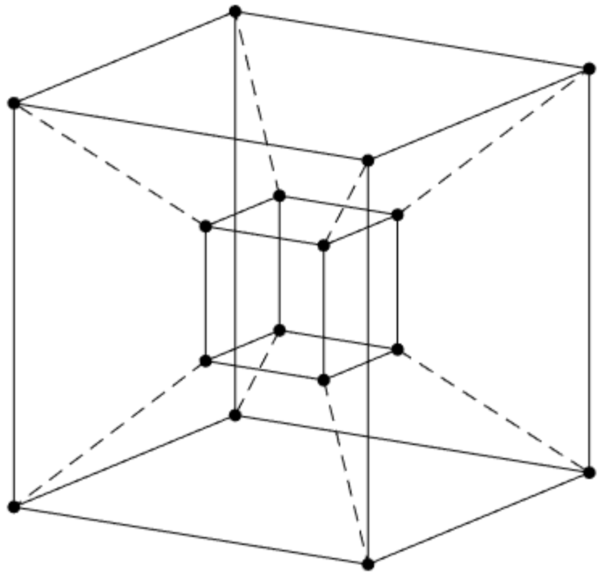
Finite Element Exterior Calculus in 4D

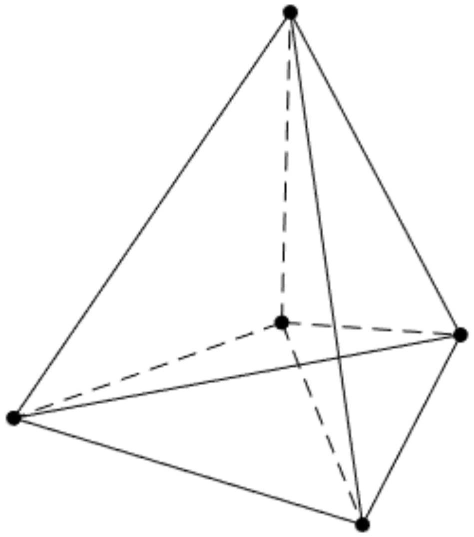
- **Objective:** identify finite-dimensional subspaces of the infinite-dimensional Sobolev spaces associated with the de Rham complex in four-dimensions

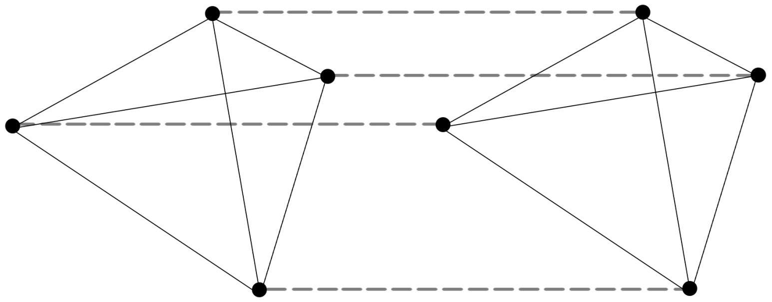
- **Characteristics of finite element spaces**
  - Conforming
  - Compatibility between elements
  - Polynomial approximation
  - Unisolvence
  - Stability (satisfies a commuting diagram property)

Finite Element Exterior Calculus in 4D

- Elements of interest
  - Tesseract (4-cube)
  - Pentatope (4-simplex)
  - Tetrahedral prism (tensor product of 3-simplex and line)

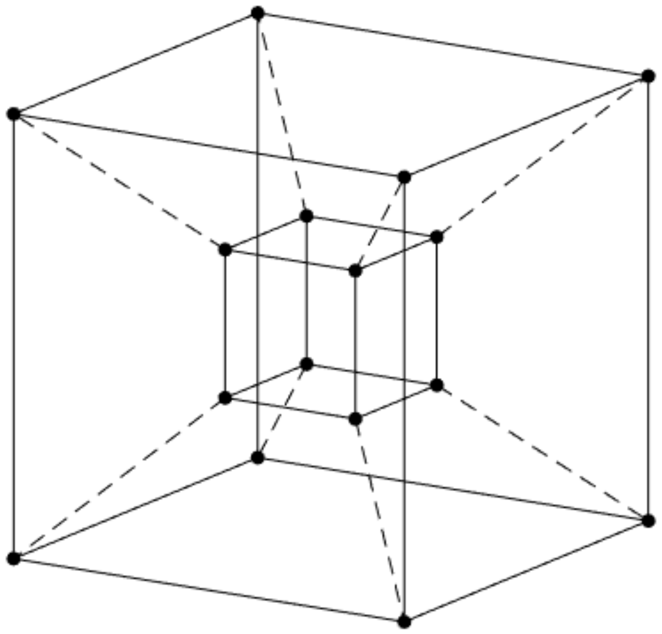






The Tesseract

- Geometric properties of the 4-cube $\mathfrak{H}^4$



	Quadrilateral $\mathfrak{H}^2$	Hexahedron $\mathfrak{H}^3$	Tesseract $\mathfrak{H}^4$
Vertices $\mathfrak{H}^0$	4	8	16
Edges $\mathfrak{H}^1$	4	8	32
Quadrilateral faces $\mathfrak{H}^2$	0	6	24
Hexahedral facets $\mathfrak{H}^3$	0	0	8

The Tesseract

- Infinite-dimensional Sobolev spaces

$$H(\mathrm{grad}, \Omega, \mathbb{R}) \xrightarrow{\mathrm{grad}} H(\mathrm{skwGrad}, \Omega, \mathbb{R}^4) \xrightarrow{\mathrm{skwGrad}} H(\mathrm{curl}, \Omega, \mathbb{K}) \xrightarrow{\mathrm{curl}} H(\mathrm{div}, \Omega, \mathbb{R}^4) \xrightarrow{\mathrm{div}} L^2(\Omega, \mathbb{R})$$

- Conforming, high-order, finite element spaces on the Tesseract

$$V_k\Lambda^0(\mathfrak{H}^4) \xrightarrow{\quad\mathrm{grad}\quad} V_k\Lambda^1(\mathfrak{H}^4) \xrightarrow{\quad\mathrm{skwGrad}\quad} V_k\Lambda^2(\mathfrak{H}^4) \xrightarrow{\quad\mathrm{curl}\quad} V_k\Lambda^3(\mathfrak{H}^4) \xrightarrow{\quad\mathrm{div}\quad} V_k\Lambda^4(\mathfrak{H}^4)$$

- What's the precise definition of the V_k spaces?

The Tesseract

- Supporting definitions

$$Q^{l,m,n,q}(x_1, x_2, x_3, x_4) = P^l(x_1)P^m(x_2)P^n(x_3)P^q(x_4),$$

$$\mathcal{L}\left(\cdot\right) : \mathbb{R}^6 \to \mathbb{K} : \mathcal{L}\left(\begin{bmatrix} w_{12} \\ w_{13} \\ w_{14} \\ w_{23} \\ w_{24} \\ w_{34} \end{bmatrix}\right) := \begin{bmatrix} 0 & w_{12} & w_{13} & w_{14} \\ -w_{12} & 0 & w_{23} & w_{24} \\ -w_{13} & -w_{23} & 0 & w_{34} \\ -w_{14} & -w_{24} & -w_{34} & 0 \end{bmatrix}$$

The Tesseract

- Finite element spaces

$$V_k\Lambda^0(\mathfrak{H}^4) := Q^{k,k,k,k}$$

$$V_k\Lambda^1(\mathfrak{H}^4) := \left[Q^{k-1,k,k,k}, Q^{k,k-1,k,k}, Q^{k,k,k-1,k}, Q^{k,k,k,k-1}\right]^T$$

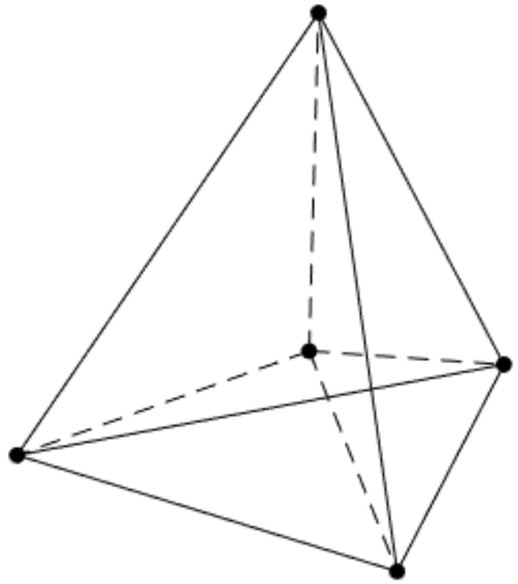
$$V_k\Lambda^2(\mathfrak{H}^4) := \mathcal{L}\left(\begin{bmatrix} Q^{k-1,k-1,k,k} \\ Q^{k-1,k,k-1,k} \\ Q^{k-1,k,k,k-1} \\ Q^{k,k-1,k-1,k} \\ Q^{k,k-1,k,k-1} \\ Q^{k,k,k-1,k-1} \end{bmatrix}\right)$$

$$V_k\Lambda^0(\mathfrak{H}^3) := \left[Q^{k,k-1,k-1,k-1}, Q^{k-1,k,k-1,k-1}, Q^{k-1,k-1,k,k-1}, Q^{k-1,k-1,k-1,k}\right]^T$$

$$V_k\Lambda^4(\mathfrak{H}^4) := Q^{k-1,k-1,k-1,k-1}$$

The Pentatope

- Geometric properties of the 4-simplex $\mathfrak{T}^4$



	Pentatope $\mathfrak{H}^4$
Vertices $\mathfrak{T}^0$	5
Edges $\mathfrak{T}^1$	10
Triangular faces $\mathfrak{T}^2$	10
Tetrahedral facets $\mathfrak{T}^3$	5

The Pentatope

- Infinite-dimensional Sobolev spaces

$$H(\mathrm{grad}, \Omega, \mathbb{R}) \xrightarrow{\mathrm{grad}} H(\mathrm{skwGrad}, \Omega, \mathbb{R}^4) \xrightarrow{\mathrm{skwGrad}} H(\mathrm{curl}, \Omega, \mathbb{K}) \xrightarrow{\mathrm{curl}} H(\mathrm{div}, \Omega, \mathbb{R}^4) \xrightarrow{\mathrm{div}} L^2(\Omega, \mathbb{R})$$

- Conforming, high-order, finite element spaces on the Pentatope

$$V_k\Lambda^0(\mathfrak{T}^4) \xrightarrow{\mathrm{grad}} V_k\Lambda^1(\mathfrak{T}^4) \xrightarrow{\mathrm{skwGrad}} V_k\Lambda^2(\mathfrak{T}^4) \xrightarrow{\mathrm{curl}} V_k\Lambda^3(\mathfrak{T}^4) \xrightarrow{\mathrm{div}} V_k\Lambda^4(\mathfrak{T}^4)$$

- What's the precise definition of the V_k spaces?

The Pentatope

- Finite element spaces

$$
\begin{aligned}
V_k\Lambda^0(\mathfrak{T}^4) &:= P^k(\mathfrak{T}^4),\\
V_k\Lambda^1(\mathfrak{T}^4) &:= (P^{k-1}(\mathfrak{T}^4))^4 \oplus \left\{p \in (\tilde{P}^k(\mathfrak{T}^4))^4 | p \cdot x = 0\right\},\\
V_k\Lambda^2(\mathfrak{T}^4) &:= \mathcal{L}\left((P^{k-1}(\mathfrak{T}^4))^6\right) \oplus \left\{B \in \mathcal{L}((\tilde{P}^k(\mathfrak{T}^4))^6) | Bx = 0\right\},\\
V_k\Lambda^3(\mathfrak{T}^4) &:= (P^{k-1}(\mathfrak{T}^4))^4 \oplus \tilde{P}^{k-1}(\mathfrak{T}^4)x,\\
V_k\Lambda^4(\mathfrak{T}^4) &:= P^{k-1}(\mathfrak{T}^4)
\end{aligned}
$$

- The second line is similar to a classical Nedelec space
- The fourth line is similar to a classical Raviart-Thomas space

The Pentatope

- Finite element spaces (restatement)

$$V_k\Lambda^2(\mathfrak{T}^4) := \mathcal{L}\left((P^{k-1}(\mathfrak{T}^4))^6\right) \oplus \tilde{P}^{k-1}(\mathfrak{T}^4)B_1 \oplus \tilde{P}^{k-1}(\mathfrak{T}^4)B_2 \oplus \tilde{P}^{k-1}(\mathfrak{T}^4)B_3 \oplus \tilde{P}^{k-1}(\mathfrak{T}^4)B_4,$$

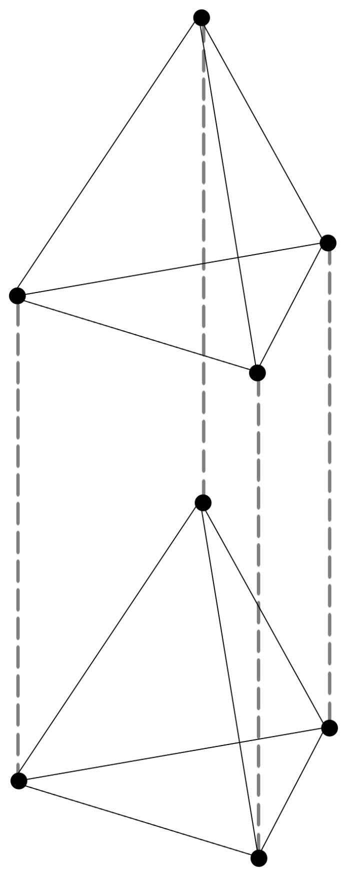
where

$$B_1 := \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & x_4 & -x_3 \\ 0 & -x_4 & 0 & x_2 \\ 0 & x_3 & -x_2 & 0 \end{bmatrix}, \qquad B_2 := \begin{bmatrix} 0 & 0 & -x_4 & x_3 \\ 0 & 0 & 0 & 0 \\ x_4 & 0 & 0 & -x_1 \\ -x_3 & 0 & x_1 & 0 \end{bmatrix},$$

$$B_3 := \begin{bmatrix} 0 & x_4 & 0 & -x_2 \\ -x_4 & 0 & 0 & x_1 \\ 0 & 0 & 0 & 0 \\ x_2 & -x_1 & 0 & 0 \end{bmatrix}, \qquad B_4 := \begin{bmatrix} 0 & -x_3 & x_2 & 0 \\ x_3 & 0 & -x_1 & 0 \\ -x_2 & x_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The Tetrahedral Prism

- Geometric properties of the prism $\mathfrak{N}^4$



	Tetrahedral Prism $\mathfrak{N}^4$
Vertices $\mathfrak{T}^0$	8
Edges $\mathfrak{T}^1$	16
Triangular faces $\mathfrak{T}^2$	8
Quadrilateral faces $\mathfrak{H}^2$	6
Tetrahedral facets $\mathfrak{T}^3$	2
Triangular prismatic facets $\mathfrak{N}^3$	4

The Tetrahedral Prism

- Infinite-dimensional Sobolev spaces

$$H(\mathrm{grad},\Omega,\mathbb{R}) \xrightarrow{\mathrm{grad}} H(\mathrm{skwGrad},\Omega,\mathbb{R}^4) \xrightarrow{\mathrm{skwGrad}} H(\mathrm{curl},\Omega,\mathbb{K}) \xrightarrow{\mathrm{curl}} H(\mathrm{div},\Omega,\mathbb{R}^4) \xrightarrow{\mathrm{div}} L^2(\Omega,\mathbb{R})$$

- Conforming, high-order, finite element spaces on the Tetrahedral Prism

$$V_k\Lambda^0(\mathfrak{N}^4) \xrightarrow{\mathrm{grad}} V_k\Lambda^1(\mathfrak{N}^4) \xrightarrow{\mathrm{skwGrad}} V_k\Lambda^2(\mathfrak{N}^4) \xrightarrow{\mathrm{curl}} V_k\Lambda^3(\mathfrak{N}^4) \xrightarrow{\mathrm{div}} V_k\Lambda^4(\mathfrak{N}^4)$$

- What's the precise definition of the V_k spaces?

The Tetrahedral Prism

- Finite element spaces

$$V_k\Lambda^0(\mathfrak{N}^4) := P^k\left(\mathfrak{T}^1\right) \otimes P^k\left(\mathfrak{T}^3\right),$$

$$V_k\Lambda^1(\mathfrak{N}^4) := P^k\left(\mathfrak{T}^1\right) \otimes \Bigg(\left\{p \,\middle|\, p \in \left[\tilde{P}^k\left(\mathfrak{T}^3\right), \tilde{P}^k\left(\mathfrak{T}^3\right), \tilde{P}^k\left(\mathfrak{T}^3\right), 0\right]^T, p \cdot x = 0\right\}$$

$$\oplus \left[P^{k-1}\left(\mathfrak{T}^3\right), P^{k-1}\left(\mathfrak{T}^3\right), P^{k-1}\left(\mathfrak{T}^3\right), 0\right]^T\Bigg) \oplus P^k\left(\mathfrak{T}^3\right) \otimes \left[0, 0, 0, P^{k-1}\left(\mathfrak{T}^1\right)\right]^T$$

The Tetrahedral Prism

- Finite element spaces

$$V_k\Lambda^2(\mathfrak{N}^4) := P^{k-1}\left(\mathfrak{T}^1\right) \otimes \left(\left\{ \begin{bmatrix} 0 & 0 & 0 & p_1 \\ 0 & 0 & 0 & p_2 \\ 0 & 0 & 0 & p_3 \\ * & * & * & 0 \end{bmatrix} \middle| \, p \in \left[\tilde{P}^k\left(\mathfrak{T}^3\right), \tilde{P}^k\left(\mathfrak{T}^3\right), \tilde{P}^k\left(\mathfrak{T}^3\right), 0 \right]^T, p \cdot x = 0 \right\} \right.$$

$$\left. \oplus \begin{bmatrix} 0 & 0 & 0 & P^{k-1}\left(\mathfrak{T}^3\right) \\ 0 & 0 & 0 & P^{k-1}\left(\mathfrak{T}^3\right) \\ 0 & 0 & 0 & P^{k-1}\left(\mathfrak{T}^3\right) \\ * & * & * & 0 \end{bmatrix} \right)$$

$$\oplus P^k\left(\mathfrak{T}^1\right) \otimes \left(\begin{bmatrix} 0 & P^{k-1}\left(\mathfrak{T}^3\right) & P^{k-1}\left(\mathfrak{T}^3\right) & 0 \\ * & 0 & P^{k-1}\left(\mathfrak{T}^3\right) & 0 \\ * & * & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \oplus \tilde{P}^{k-1}\left(\mathfrak{T}^3\right) \begin{bmatrix} 0 & x_3 & -x_2 & 0 \\ * & 0 & x_1 & 0 \\ * & * & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right)$$

The Tetrahedral Prism

- Finite element spaces

$$V_k\Lambda^3(\mathfrak{N}^4) := P^{k-1}\left(\mathfrak{T}^3\right) \otimes \left[0,0,0,P^k\left(\mathfrak{T}^1\right)\right]^T$$
$$\oplus\, P^{k-1}\left(\mathfrak{T}^1\right) \otimes \left(\left[P^{k-1}\left(\mathfrak{T}^3\right), P^{k-1}\left(\mathfrak{T}^3\right), P^{k-1}\left(\mathfrak{T}^3\right), 0\right]^T \oplus \tilde{P}^{k-1}\left(\mathfrak{T}^3\right)[x_1,x_2,x_3,0]^T \right),$$

$$V_k\Lambda^4(\mathfrak{N}^4) := P^{k-1}\left(\mathfrak{T}^1\right) \otimes P^{k-1}\left(\mathfrak{T}^3\right)$$

Finite Element Exterior Calculus in 4D

- Practical considerations
  - Additional applications
  - Polynomial basis functions
  - Degrees of freedom
  - Bubble spaces
  - Actual proofs

- All these concepts and proofs of unisolvence for the degrees of freedom are described in a pair of papers.

Finite Element Exterior Calculus in 4D

- Relevant papers

Computers and Mathematics with Applications 166 (2024) 198–223

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ELSEVIER

Conforming finite element function spaces in four dimensions, part I: Foundational principles and the tesseract

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Finite Element Exterior Calculus in 4D

- Relevant papers

Computers and Mathematics with Applications 167 (2024) 21–53

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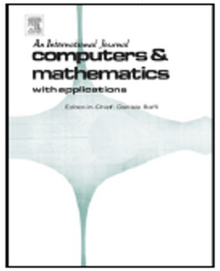

ELSEVIER

Conforming finite element function spaces in four dimensions, part II: The pentatope and tetrahedral prism

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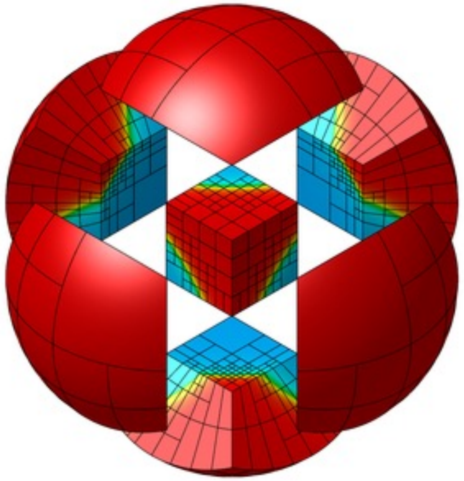
Numerical Experiments

Numerical Experiments

- **MFEM**
- “Free, lightweight, scalable C++ library for modular finite element methods”



- **Advantages**
  - A 4D branch of MFEM already exists
  - Developed by Martin Neumüller and Andreas Schafelner
  - Simple implementation of H1-conforming basis functions (high order)
  - Low-order H(div)-conforming basis functions

- **Disadvantages**
  - Non-working implementation of H(div)-conforming basis functions (high order)
  - Non-working bisection-based mesh refinement algorithm

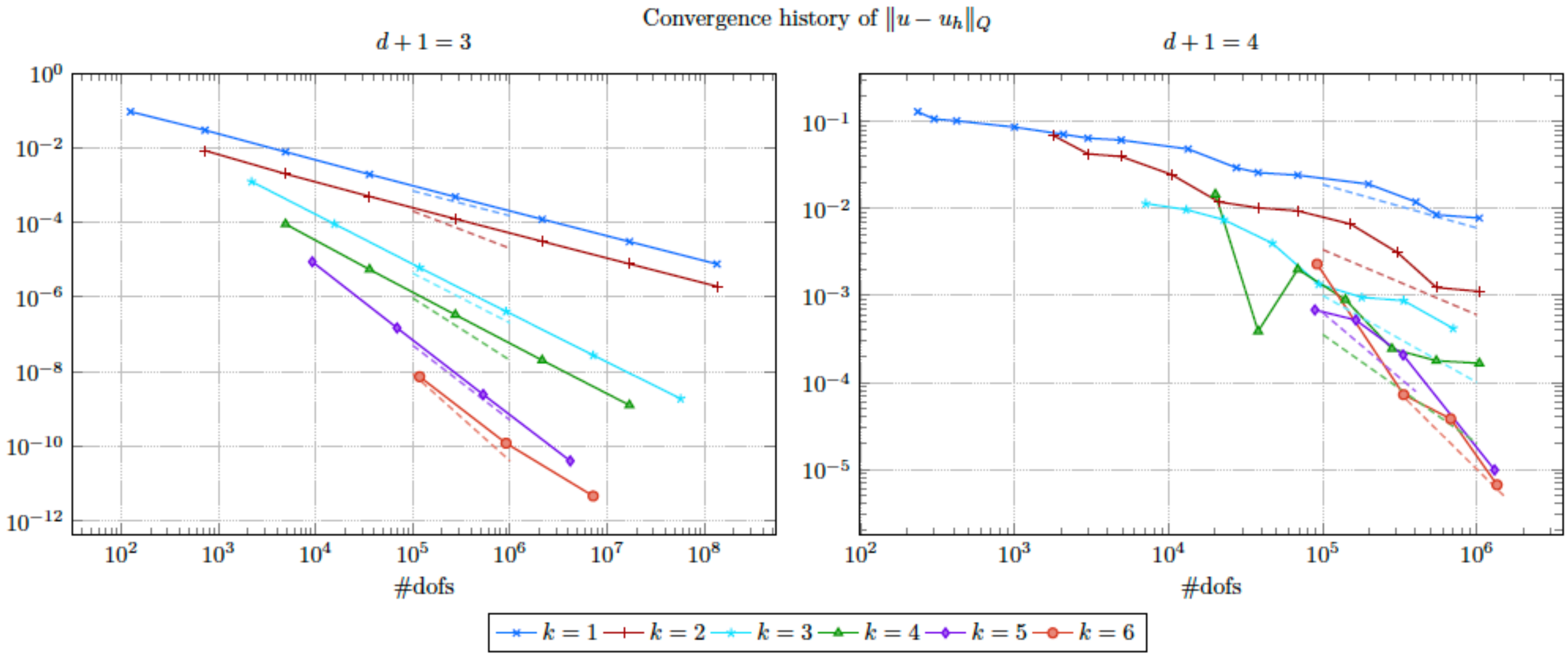
Numerical Experiments

- **Key resources:**

1. Andreas Schafelner, “Space-Time Finite Element Methods for Parabolic Initial-Boundary Problems,” Masters Thesis, Johannes Kepler University Linz, 2017
2. Andreas Schafelner, “Space-Time Finite Element Methods,” PhD Thesis, Johannes Kepler University Linz, 2021

Numerical Experiments

- **L_2 solution error for parabolic diffusion problem**



Numerical Experiments

- **Problem:** convergence rates of H1-conforming basis functions are not convincing

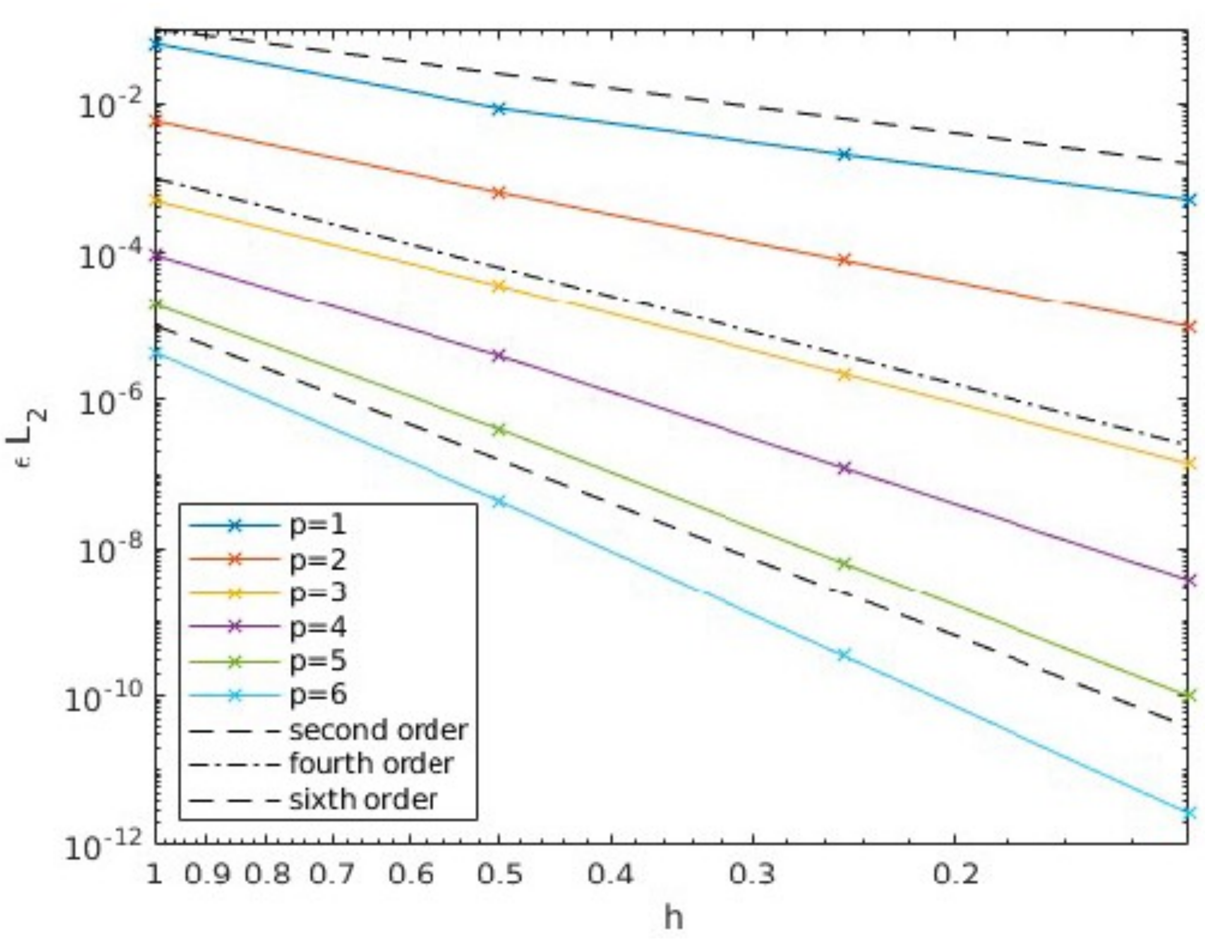
- **Solution I:**
  - Abandon bisection-based mesh refinement approach
  - Generate grid family separately
  - Develop a sequence of refined grids using 4D Delaunay mesh generation
  - Grids are quasi-uniform, with grid points laid out in a rectilinear array
  - Resulting grids are very similar to Coxeter-Freudenthal-Kuhn Triangulations

Numerical Experiments

- **Solution II:**
  - Implement high-order H1- and H(div)-conforming basis functions based on papers of Nigam and Williams
  - Implement correct orientations for 4-simplexes (pentatopic elements)
  - Implement correct orientations for 3-simplexes (tetrahedral facets)
  - Remove other bugs, including array-sizing errors

Numerical Experiments

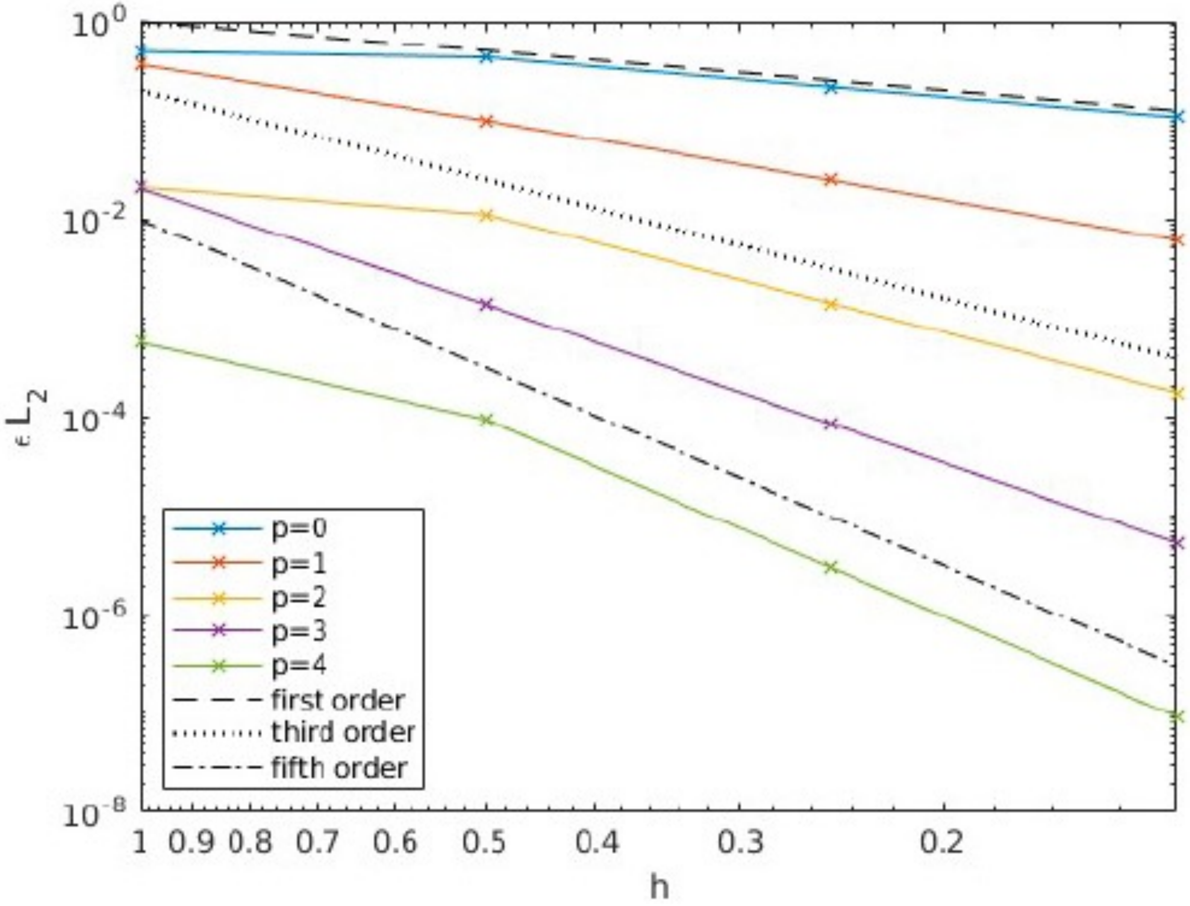
- **L_2 error for projection of transcendental scalar function, H1-conforming basis**



order	mesh size h	L^2 error	rate
$p=1$	1	6.38E-2	-
	0.5	8.59E-3	2.8920
	0.25	2.069E-3	2.0553
	0.125	5.12E-4	2.0139
$p=2$	1	5.85E-3	-
	0.5	6.39E-4	3.1943
	0.25	7.81E-05	3.0344
	0.125	9.70E-06	3.0088
$p=3$	1	4.96E-4	-
	0.5	3.53E-05	3.8136
	0.25	2.21E-06	4.0000
	0.125	1.38E-07	4.0000
$p=4$	1	9.23E-05	-
	0.5	3.88E-06	4.5709
	0.25	1.21E-07	5.0000
	0.125	3.79E-09	4.9999
$p=5$	1	1.94E-05	-
	0.5	4.04E-07	5.5881
	0.25	6.31E-09	5.9999
	0.125	9.86E-11	6.0000
$p=6$	1	4.33E-06	-
	0.5	4.38E-08	6.6295
	0.25	3.42E-10	6.9989
	0.125	2.67E-12	7.0000

Numerical Experiments

- **L_2 error for projection of transcendental vector function, H(div)-conforming basis**



order	mesh size h	L^2 error	rate
$p=0$	1	4.89E-1	-
	0.5	4.27E-1	0.1934
	0.25	2.14E-1	0.9946
	0.125	1.07E-1	0.9986
$p=1$	1	3.614E-1	-
	0.5	9.92E-2	1.8642
	0.25	2.45E-2	2.0183
	0.125	6.11E-3	2.0040
$p=2$	1	0.0210	-
	0.5	1.11E-2	0.9121
	0.25	1.41E-03	2.9859
	0.125	1.7E-04	2.9964
$p=3$	1	2.07E-02	-
	0.5	1.38E-03	3.9073
	0.25	8.52E-05	4.0158
	0.125	5.31E-06	4.0032
$p=4$	1	5.63E-04	-
	0.5	9.50E-05	2.5685
	0.25	3.00E-06	4.9861
	0.125	9.39E-08	4.9964

Next Steps

- **Future work**
  - Test H1-conforming basis functions on real problems
  - Test H(div)-conforming basis functions on real problems
  - Implement H(skwGrad)- and H(curl)-conforming basis functions
  - Simulate unsteady, inhomogeneous Maxwell’s equations

- Currently, PhD student Patrick Saber is working on this project

Questions?