# Computation and Reduced Order Modelling of Periodic Flows

## with applications to lift control of hydrofoils

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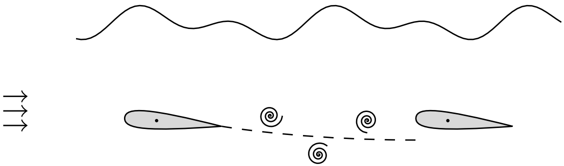
October 26th, 2023

# Hydrofoil craft could be an alternative green way of transportation

- Comfortable
- Low resistance
- Requires little infrastructure
- Speeds up to 70 km/h (44 m/h)

# Need for improved lift control system



We need fast computation of lift

- Actuator design
- System control

# Introduction to projection-based Reduced Order Models

**Full order system of equations**

$$\mathbf{A}\boldsymbol{\phi} - \mathbf{b} = \mathbf{0}$$

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**We can approximate the solution**

$$\boldsymbol{\phi} \approx \mathbf{V}\hat{\boldsymbol{\phi}}$$

$\mathbf{V}$ is a solution basis (POD, Greedy)

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$\mathbf{V}$ is a solution basis (POD, Greedy)

**Reduced system of equations**

Approximate $\phi$

$$\mathbf{A}\mathbf{V}\hat{\boldsymbol{\phi}} - \mathbf{b} = \mathbf{0}$$

Galerkin projection

$$\mathbf{V}^T\mathbf{A}\mathbf{V}\hat{\boldsymbol{\phi}} - \mathbf{V}\mathbf{b} = \mathbf{V}\mathbf{0}$$

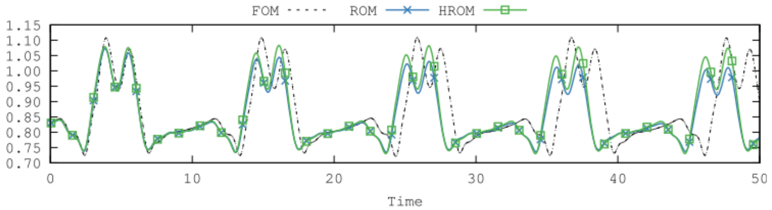
Reduced system of equations depending on $\hat{\phi}$ size

$$\hat{\mathbf{A}}\hat{\boldsymbol{\phi}} - \hat{\mathbf{b}} = \mathbf{0}$$

# We want to avoid integrating over a large time domain

- To reduced computational requirements
- To avoid an unwanted phase shift in time, see for example [1]

velocity [1]

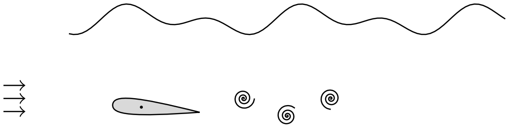



[1] R. Reyes and R. Codina (May 2020). "Projection-based reduced order models for flow problems: A variational multiscale approach". In: *Computer Methods in Applied Mechanics and Engineering* 363

# The remainder of the presentation

**Our approach**

- In the full order model, we assume the flow to be periodic
- We only simulate one period
- The system becomes a boundary value problem
- We create a time periodic reduced order model



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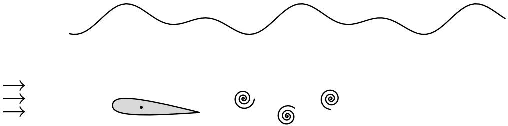
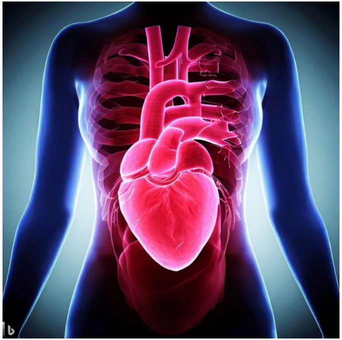
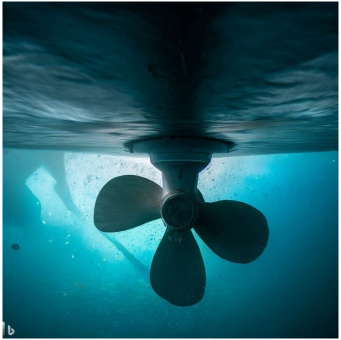


**delFI** - Delft Finite-element and Isogeometric-analysis

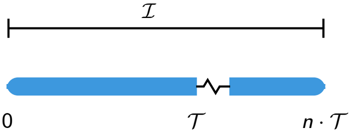
# Periodic flows are omnipresent in our world

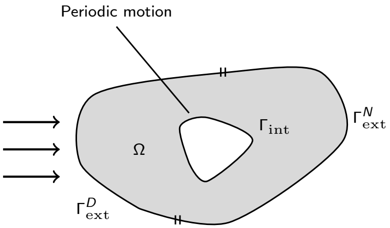




AI generated

# The model problem of periodic flows



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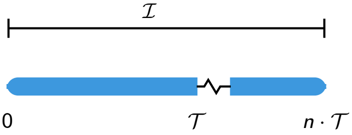
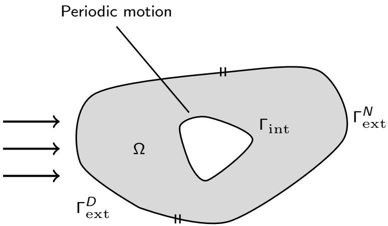
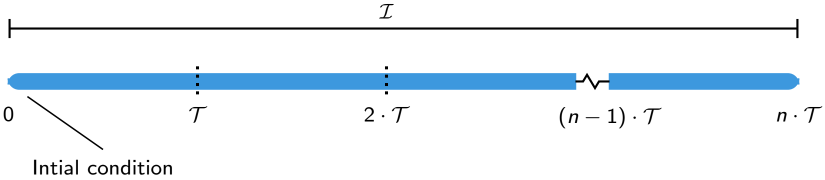



$$
\begin{aligned}
\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p - \nabla \cdot (2\nu \nabla^s \mathbf{u}) &= \mathbf{f} && \text{in} \quad \Omega, \\
\nabla \cdot \mathbf{u} &= 0 && \text{in} \quad \Omega, \\
\mathbf{u} &= \mathbf{g}_{\mathrm{int}}(t) && \text{in} \quad \Gamma_{\mathrm{int}}, \\
\mathbf{u} &= \mathbf{g}_{\mathrm{ext}} && \text{in} \quad \Gamma^D_{\mathrm{ext}}, \\
-p\mathbf{n} + \nu \nabla \mathbf{u} \cdot \mathbf{n} + u_{\mathrm{n}}^- \mathbf{u} &= \mathbf{0} && \text{in} \quad \Gamma^N_{\mathrm{ext}}, \\
\mathbf{u}(\cdot, 0) &= \mathbf{u}_0 && \text{in} \quad \Omega.
\end{aligned}
$$

# Periodic time domain



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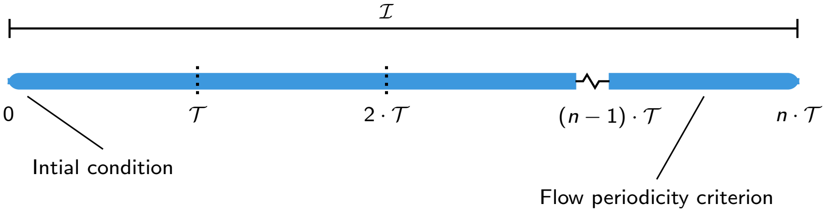

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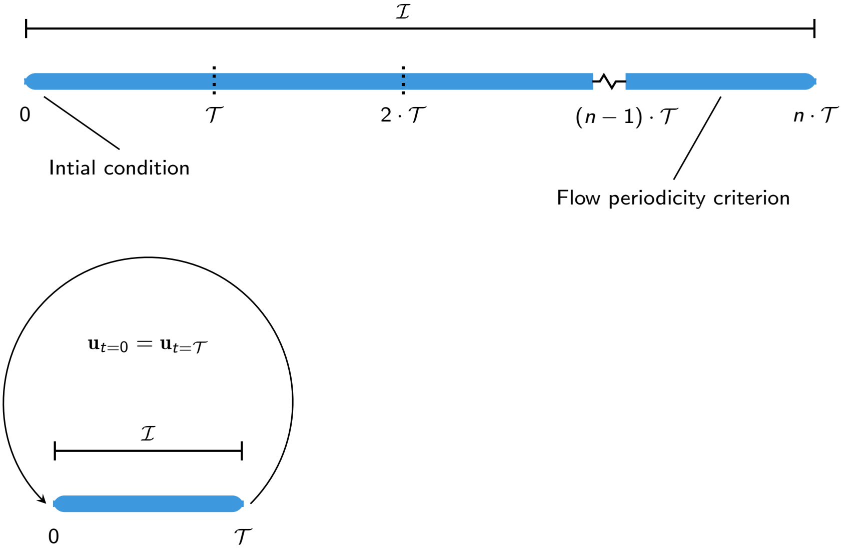
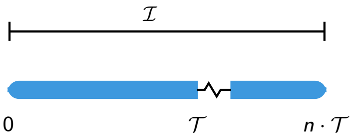
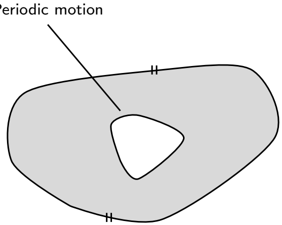


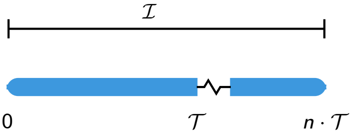
# The space-time domain

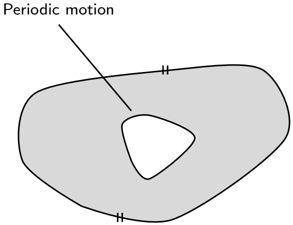


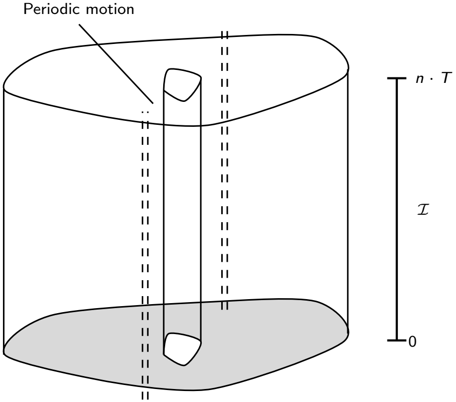




Discretize with: Semi-discretisation

# The space-time domain



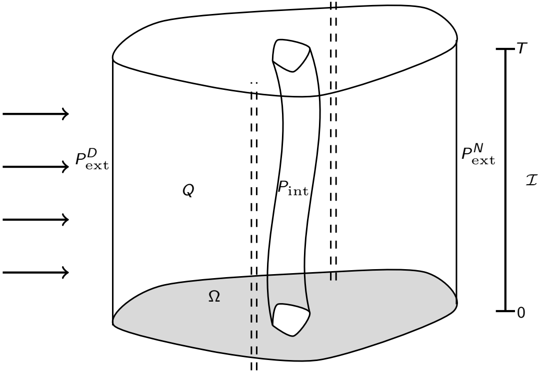




Discretize with: Semi-discretisation

Space-time discretisation

# The space-time domain with enforced temporal periodicity




**Initial value problem becomes a boundary value problem**

$$\mathbf{u}(\cdot, 0) = \mathbf{u}_0 \quad \text{in} \quad \Omega$$

$$\Downarrow$$

$$\mathbf{u}(\cdot, 0) = \mathbf{u}(\cdot, T) \quad \text{in} \quad \Omega$$

# Two features of the weak model problem

$$
\begin{aligned}
& B_{\mathrm{GAL}}\left(\boldsymbol{U}^h, \boldsymbol{W}^h\right) + B_{\mathrm{PT}}\left(\boldsymbol{U}^h, \boldsymbol{W}^h\right) \\
& + B_{\mathrm{VMS}}\left(\boldsymbol{U}^h, \boldsymbol{W}^h\right) + B_{\mathrm{WBC}}\left(\boldsymbol{U}^h, \boldsymbol{W}^h\right) = L\left(\boldsymbol{W}^h\right),
\end{aligned}
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\end{aligned}
$$

*where*

$$
\begin{aligned}
B_{\mathrm{GAL}}(\boldsymbol{U}, \boldsymbol{W}) = & \quad (\mathbf{w}, \hat{\mathbf{u}} \cdot \nabla_{\hat{\mathbf{x}}} \mathbf{u})_Q + (\nabla \cdot \mathbf{w}, p)_Q \\
& + (\nabla \mathbf{w}, \nu \nabla \mathbf{u})_Q + (q, \nabla \cdot \mathbf{u})_Q - (\mathbf{w}, u_{\mathrm{n}}^{-} \mathbf{u})_{P_{\mathrm{ext}}^N} \\
B_{\mathrm{PT}}\left(\boldsymbol{U}^h, \boldsymbol{W}^h\right) = & \quad \left(\mathbf{w}^h, \partial_\theta \mathbf{u}^h\right)_Q + \frac{1}{a^2}\left(q^h, \partial_\theta p^h\right)_Q, \\
B_{\mathrm{VMS}}\left(\boldsymbol{U}^h, \boldsymbol{W}^h\right) = & \quad -\left(\nabla_{\hat{\mathbf{x}}} \mathbf{w}^h, \mathbf{u}' \otimes \hat{\mathbf{u}}^h\right)_{\tilde{Q}} - \left(\nabla \mathbf{w}^h, \mathbf{u}^h \otimes \mathbf{u}'\right)_{\tilde{Q}} \\
& - \left(\nabla \mathbf{w}^h, \mathbf{u}' \otimes \mathbf{u}'\right)_{\tilde{Q}} - \left(\nabla q^h, \mathbf{u}'\right)_{\tilde{Q}} - \left(\nabla \cdot \mathbf{w}^h, p'\right)_{\tilde{Q}}, \\
B_{\mathrm{WBC}}\left(\boldsymbol{U}^h, \boldsymbol{W}^h\right) = & \quad \left(\mathbf{w}^h, p^h \mathbf{n} - \nu \nabla \mathbf{u}^h \cdot \mathbf{n}\right)_{P_{\mathrm{int}}} + \left(\nu \nabla \mathbf{w}^h \cdot \mathbf{n} - q^h \mathbf{n}, \mathbf{u}^h - \mathbf{g}\right)_{P_{\mathrm{int}}} \\
& + \left(\mathbf{w}^h \tau_b, \mathbf{u}^h - \mathbf{g}\right)_{P_{\mathrm{int}}}.
\end{aligned}
$$

# Two features of the weak model problem

$$B_{\mathrm{GAL}}\left(\boldsymbol{U}, \boldsymbol{W}\right) \qquad (\mathbf{w}, \hat{\mathbf{u}} \cdot \nabla_{\hat{\mathbf{x}}} \mathbf{u})_Q$$

$$B_{\mathrm{VMS}}\left(\boldsymbol{U}^h, \boldsymbol{W}^h\right) \qquad -\left(\nabla_{\hat{\mathbf{x}}} \mathbf{w}^h, \mathbf{u}' \otimes \hat{\mathbf{u}}^h\right)_{\tilde{Q}}$$

**Space-time velocity**

$$\hat{\mathbf{x}} = [\mathbf{x}^T \;\; t]^T$$

$$\hat{\mathbf{u}} = [\mathbf{u}^T \;\; 1]^T$$

$$\Downarrow$$

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = \hat{\mathbf{u}} \cdot \nabla_{\hat{\mathbf{x}}} \mathbf{u},$$

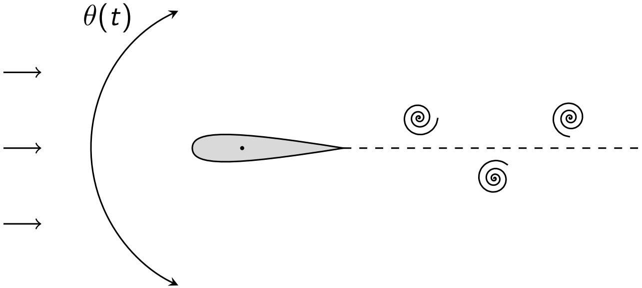
# Two features of the weak model problem

$$B_{\mathrm{PT}}\left(\boldsymbol{U}^h, \boldsymbol{W}^h\right) = \left(\mathbf{w}^h, \partial_\theta \mathbf{u}^h\right)_Q + \frac{1}{a^2}\left(q^h, \partial_\theta p^h\right)_Q,$$

**Pseudo compressibility**

Artificial speed of sound: $a$

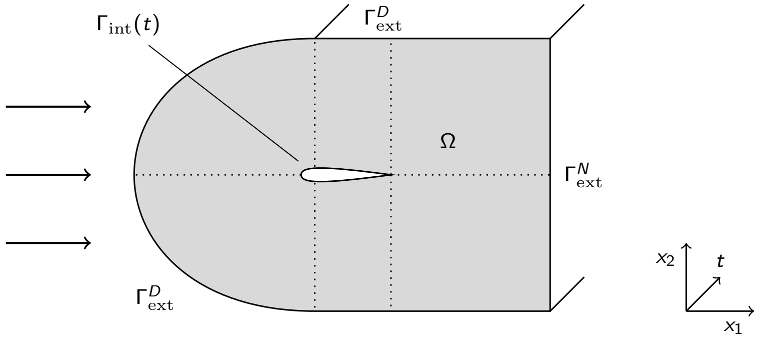
# Numerical experiment: pitching hydrofoil

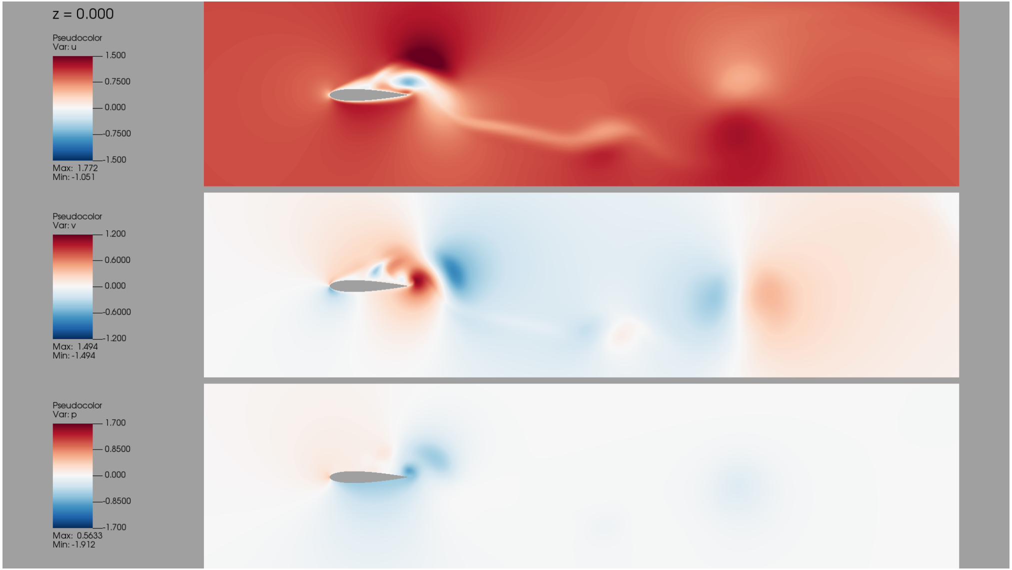



$$\theta\left(t\right) = \theta_a \sin\left(2\pi \frac{t}{T}\right)$$

$$Re = 1000$$

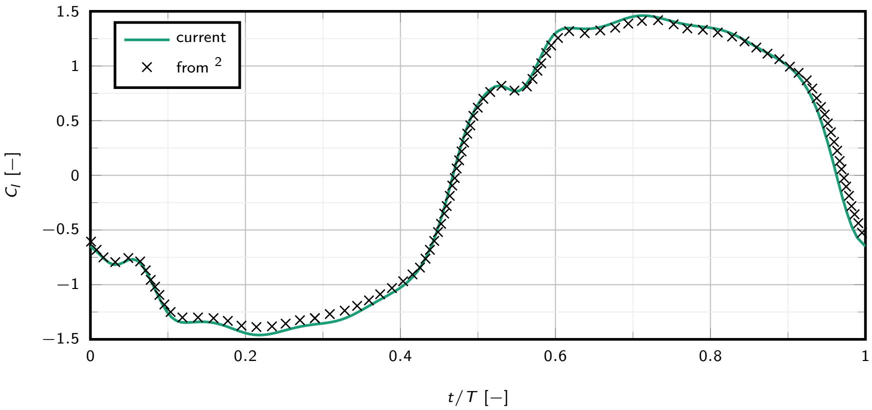
# The domain is discretized using isogeometric analysis






Slices in space-time domain, $Re = 1000$, $\theta_a = 23^\circ$, $T = 8.33s$

# Conservative traction evaluation




[2] T. Kinsey and G. Dumas (2008). "Parametric study of an oscillating airfoil in a power-extraction regime". In: *AIAA Journal* 46.6, pp. 1318–1330

# POD-Galerkin Reduced Order Model

**System of equations of full order model**

$$\begin{bmatrix} \mathbf{A}_u + \mathbf{H}_{wu}(\mathbf{u}^h) & \mathbf{B}_w + \mathbf{H}_{wp}(\mathbf{u}^h) \\ \mathbf{B}_q & \mathbf{A}_p \end{bmatrix} \begin{bmatrix} \phi \\ \psi \end{bmatrix} = \begin{bmatrix} \mathbf{b}_w \\ \mathbf{b}_q \end{bmatrix} \tag{6}$$

# POD-Galerkin Reduced Order Model

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**Reduced system of equations**

Combined velocity and pressure basis:

$$\mathbf{u} = \mathbf{V}_u \hat{\phi} \tag{7a}$$

$$p = \mathbf{V}_p \hat{\psi} \tag{7b}$$

After Galerkin projection:

$$\begin{bmatrix} \hat{\mathbf{A}}_u + \hat{\mathbf{H}}_{wu}(\mathbf{u}) & \hat{\mathbf{B}}_w + \hat{\mathbf{H}}_{wp}(\mathbf{u}) \\ \hat{\mathbf{B}}_q & \hat{\mathbf{A}}_p \end{bmatrix} \begin{bmatrix} \hat{\phi} \\ \hat{\psi} \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{b}}_w \\ \hat{\mathbf{b}}_q \end{bmatrix} \tag{8}$$

**Components**

$$\hat{\mathbf{A}}_u = \mathbf{V}_u^T \mathbf{A}_u \mathbf{V}_u \tag{9a}$$

$$\hat{\mathbf{A}}_p = \mathbf{V}_p^T \mathbf{A}_u \mathbf{V}_p \tag{9b}$$

$$\hat{\mathbf{B}}_w = \mathbf{V}_u^T \mathbf{B}_w \mathbf{V}_p \tag{9c}$$

$$\hat{\mathbf{B}}_q = \mathbf{V}_p^T \mathbf{B}_q \mathbf{V}_u \tag{9d}$$

$$\hat{\mathbf{H}}_{wu}(\mathbf{u}) = \mathbf{V}_u^T \hat{\mathbf{H}}_{wu}(\mathbf{V}_u^T \hat{\phi}) \tag{9e}$$

$$\hat{\mathbf{H}}_{wp}(\mathbf{u}) = \mathbf{V}_u^T \hat{\mathbf{H}}_{wp}(\mathbf{V}_u^T \hat{\phi}) \tag{9f}$$

# First test case

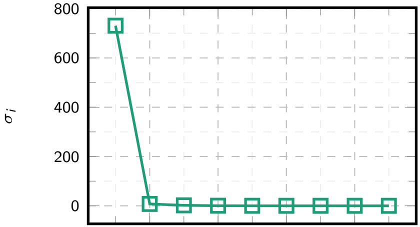
**Vary Strouhall number (speed-up $\approx$ 40)**

- Vary temporal period between 7.5 $s$ and 9.4 $s$ ($St_A$: 0.011 : 0.013)

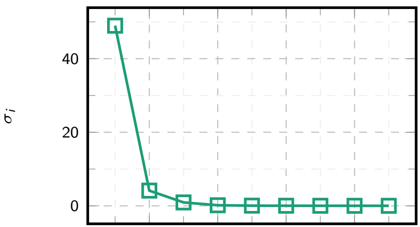
$$St_A = \frac{f \ h_a}{U_\infty} \tag{10}$$

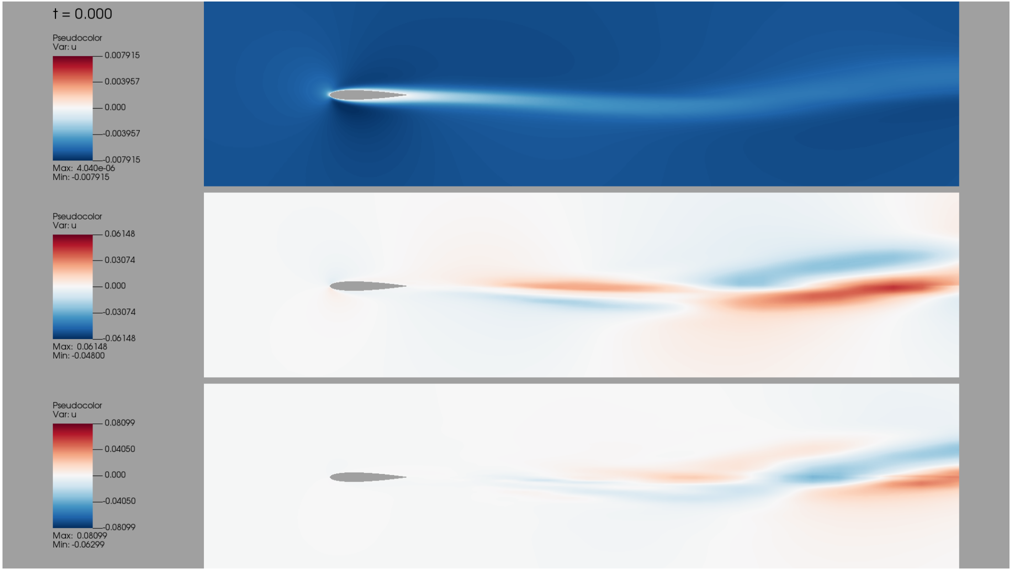
- Number of samples: 25
- Fast decay of singular values of POD

Velocity: $EF_{0.99999}$ : 22/25

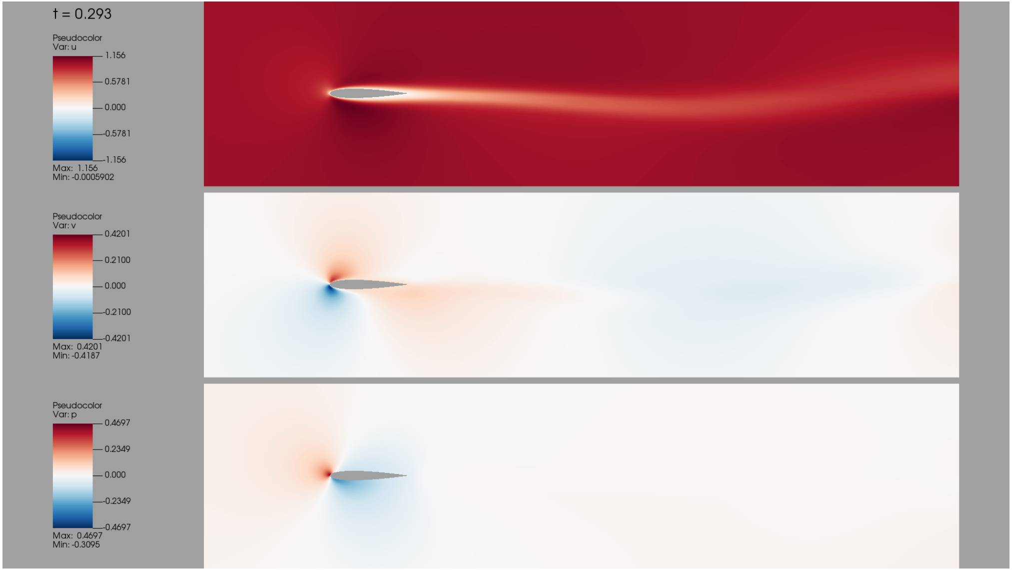



Pressure: $EF_{0.99999}$ : 22/25



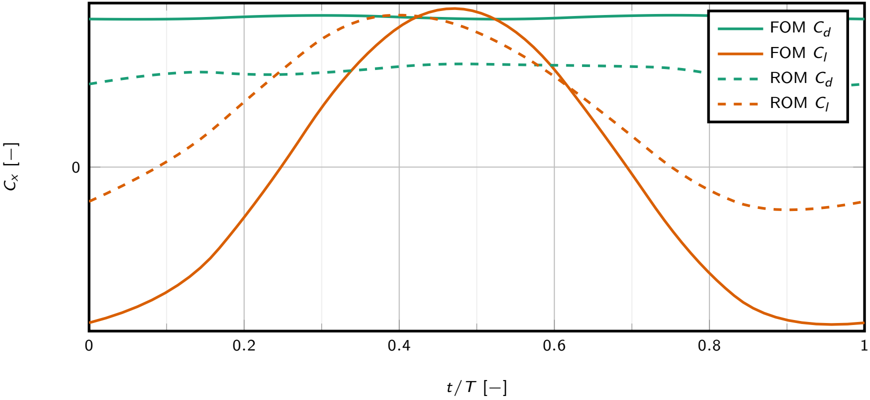



First three modes of $u_1$-basis, first part of $\mathbf{V}_u$




ROM solution. Relative errors: $e_{L_1} = 0.8\%$, $e_{L_2} = 0.7\%$, $e_{L_\infty} = 1.4\%$

# Comparing forces



# Wrap-up

**We have seen:**

Full order model:

- Time-periodic space-time method [3]
- One period as time domain
- Matching results with literature

Reduced order model, first test case shows:

- Small error for flow fields
- Large error for forces

**Future work:**

- More challenging test cases
- Improve quality of forces
- Finish implementation of hyper-reduction

Github: @JacobLotz

E-mail: j.e.lotz@tudelft.nl

[3] J. E. Lotz, M. F. P. ten Eikelder, and I Akkerman (2023). "Space-time computations of exact time-periodic flows past hydrofoils". In: *Under review at Computers & Fluids, arXiv:2211.10964*, pp. 1–19

# Two features of the weak model problem

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\begin{aligned}
&B_{\mathrm{GAL}}\left(\boldsymbol{U}^h,\boldsymbol{W}^h\right)+B_{\mathrm{PT}}\left(\boldsymbol{U}^h,\boldsymbol{W}^h\right)\\
&+B_{\mathrm{VMS}}\left(\boldsymbol{U}^h,\boldsymbol{W}^h\right)+B_{\mathrm{WBC}}\left(\boldsymbol{U}^h,\boldsymbol{W}^h\right)=L\left(\boldsymbol{W}^h\right),
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&\quad-\left(\nabla\mathbf{w}^h,\mathbf{u}'\otimes\mathbf{u}'\right)_{\tilde{Q}}-\left(\nabla q^h,\mathbf{u}'\right)_{\tilde{Q}}-\left(\nabla\cdot\mathbf{w}^h,p'\right)_{\tilde{Q}},\\
B_{\mathrm{WBC}}\left(\boldsymbol{U}^h,\boldsymbol{W}^h\right) &= \left(\mathbf{w}^h,p^h\mathbf{n}-\nu\nabla\mathbf{u}^h\cdot\mathbf{n}\right)_{P_{\mathrm{int}}}+\left(\nu\nabla\mathbf{w}^h\cdot\mathbf{n}-q^h\mathbf{n},\mathbf{u}^h-\mathbf{g}\right)_{P_{\mathrm{int}}}\\
&\quad+\left(\mathbf{w}^h\tau_b,\mathbf{u}^h-\mathbf{g}\right)_{P_{\mathrm{int}}}.
\end{aligned}
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**Pseudo compressibility**

Artificial speed of sound: $a$