Dissipation-Based Entropy Stabilization for Slope-Limited Discontinuous Galerkin Approximations of Hyperbolic Problems

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Problem statement

A general hyperbolic problem

$$\frac{\partial u}{\partial t} + \nabla \cdot \mathbf{f}(u) = 0 \quad \text{in } \Omega \times \mathbb{R}^+$$
$$u = u_0 \quad \text{in } \Omega \times \{0\}$$

- Euler equations: $u = (\rho, \rho \mathbf{v}, \rho E)^T$
- Preservation of invariant domains: There is a convex set $\mathcal{G}$ such that

$$u(\mathbf{x}, t) \in \mathcal{G} \quad \forall \mathbf{x} \in \Omega \, \forall t \geq 0$$

- Entropy stability: The vanishing viscosity solution satisfies

$$\frac{\partial \eta(u)}{\partial t} + \nabla \cdot \mathbf{q}(u) \leq 0 \quad \text{in } \Omega \times \mathbb{R}^+$$

Sequential slope limiting for systems

- Nonlinear system: $u = (\varrho, \varrho\phi_1, \ldots, \varrho\phi_{m-1})$
- Limit density-like variable on element K_i

$$\varrho^*_{ih} = \overline{\varrho}_i + \alpha^{\varrho}_i(\varrho_{ih} - \overline{\varrho}_i)$$

- Limit conserved products[1]

$$(\varrho\phi)^*_{ih} = \varrho^*_{ih}\overline{\phi}_i + \alpha^{\phi}_i((\varrho\phi)_{ih} - \varrho^*_{ih}\overline{\phi}_i), \quad \overline{\phi} := \frac{\overline{(\varrho\phi)_i}}{\overline{\varrho}_i},$$

- Euler equations: IDP limiting

$$u^{\mathrm{IDP}}_{ih} = \overline{u}_i + \alpha^{\mathrm{IDP}}_i(u^*_{ih} - \overline{u}_i) \in \mathcal{G}$$

- Cell averages after SSP-RK-Euler stage convex combination of admissible states, i.e., **invariant domain preservation**

Good news!

This limiter is very easy to implement in MFEM!

[1] Dobrev, Kolev, Kuzmin, Rieben, and Tomov (2018)

Entropy stabilization

- Enforce local semi-discrete entropy inequality

$$\int_{K_i} \frac{\partial \eta(u_{ih})}{\partial t} \mathrm{d}\mathbf{x} + \sum_j \int_{K_i \cap K_j} Q_{ij} \cdot \mathbf{n}_{ij} \mathrm{d}\mathbf{s} \leq 0$$

- Extend standard DG formulation by adding[2]

$$\nu_i D_i(v_{ih}, w_{ih}) = \nu_i \int_{K_i} (v_{ih} - \overline{v}_i)(w_{ih} - \overline{w}_i) \mathrm{d}\mathbf{x}$$

- Entropy viscosity coefficient $\nu_i \geq 0$ chosen such that[3]

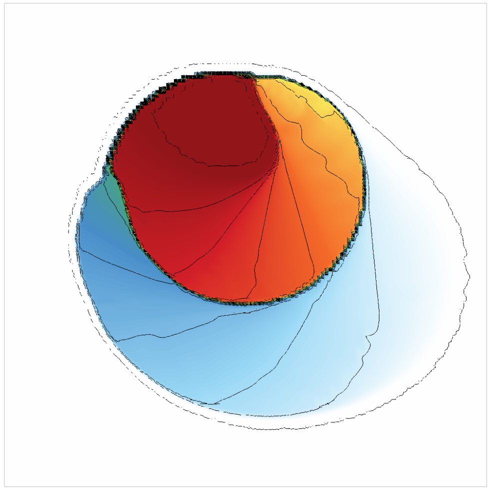
$$\int_{K_i} \nabla(v_{ih} - \overline{v}_i) : \mathbf{f}(u_{ih}) \mathrm{d}\mathbf{x} - \nu_i D_i(v_{ih}, v_{ih}) \leq \sum_j \int_{K_i \cap K_j} \big(\boldsymbol{\psi}(u_{ih}) - \boldsymbol{\psi}(\overline{u}_i)\big) \cdot \mathbf{n}_{ij} \mathrm{d}\mathbf{s}$$

- **Importantly**, evolution equations for the cell averages are left unchanged

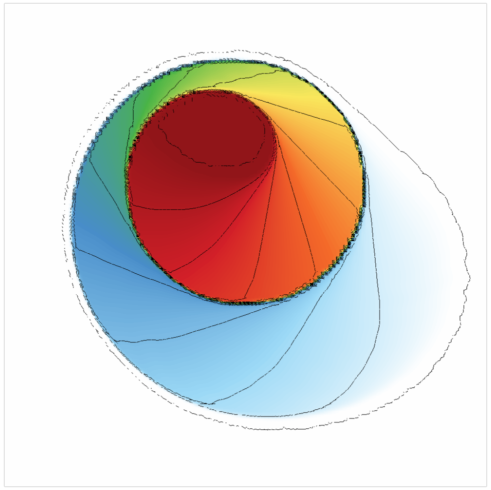
[2] Abgrall (2018)
[3] Kuzmin and Hajduk (2023)

Numerical examples

- Scalar 2D KPP problem

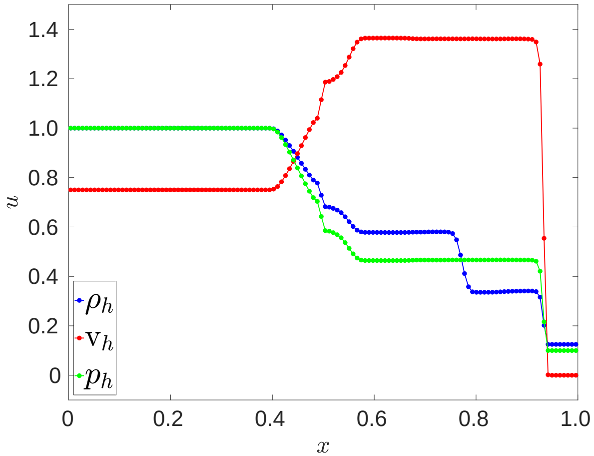


Limiter

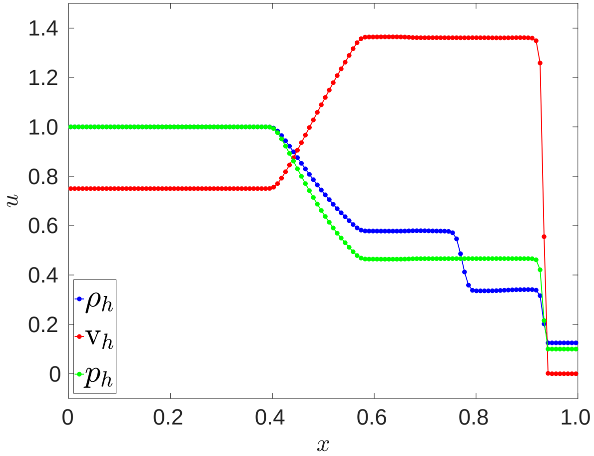


Limiter + Entropy stab.

- Modified Sod shock tube problem



Limiter



Limiter + Entropy stab.

References

- V. Dobrev, T. Kolev, D. Kuzmin, R. Rieben and V. Tomov, Sequential limiting in continuous and discontinuous Galerkin methods for the Euler equations. *J. Comput. Phys.* **356** (2018) 372-390
- R. Abgrall, A general framework to construct schemes satisfying additional conservation relations. Application to entropy conservative and entropy dissipative schemes. *J. Comput. Phys.* **372** (2018) 640-666
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