Semi-Lagrangian characteristic reconstruction and projection for transport under incompressible velocity fields

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first.
further.
forward.

Scalar Advection

Forward Euler, Upwind Scheme

$$\frac{\partial \phi}{\partial t} + \mathbf{u} \cdot \nabla \phi = 0 \quad \longrightarrow \quad \phi^{n+1} = \phi^n + \frac{\mathbf{u}\Delta t}{\Delta x}\left(\phi_i^n - \phi_{i-1}^n\right)$$

Lagrangian Form

$$\frac{D\phi}{Dt} = 0$$

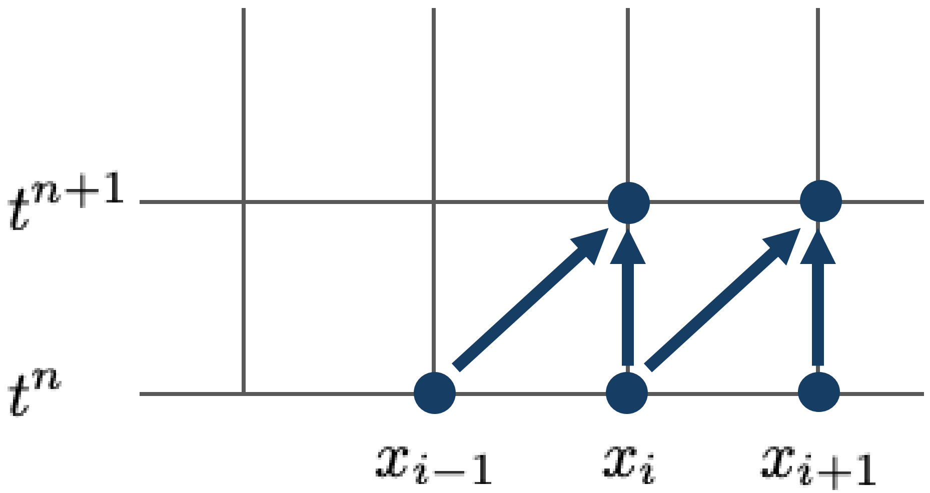
Constant along characteristics

$$\frac{Dx}{Dt} = \mathbf{u}$$

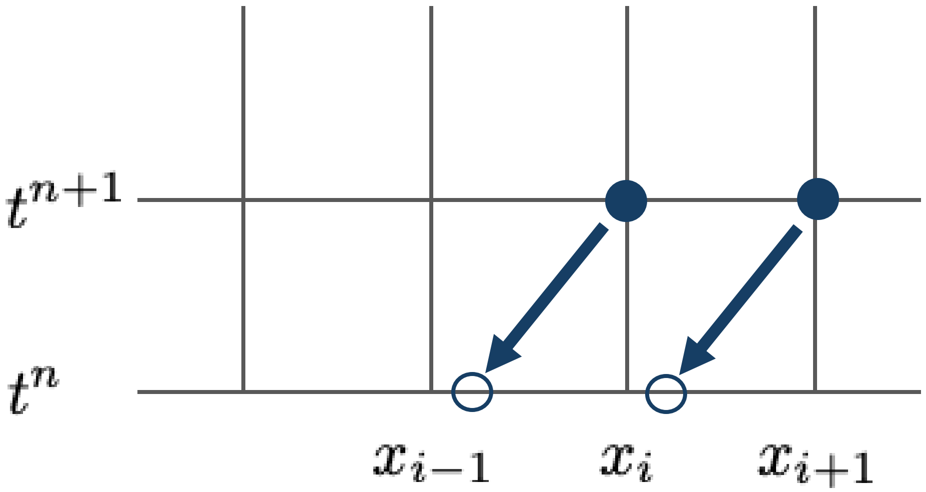
Characteristic velocity

Semi-Lagrangian Method

Forward Euler, Upwind



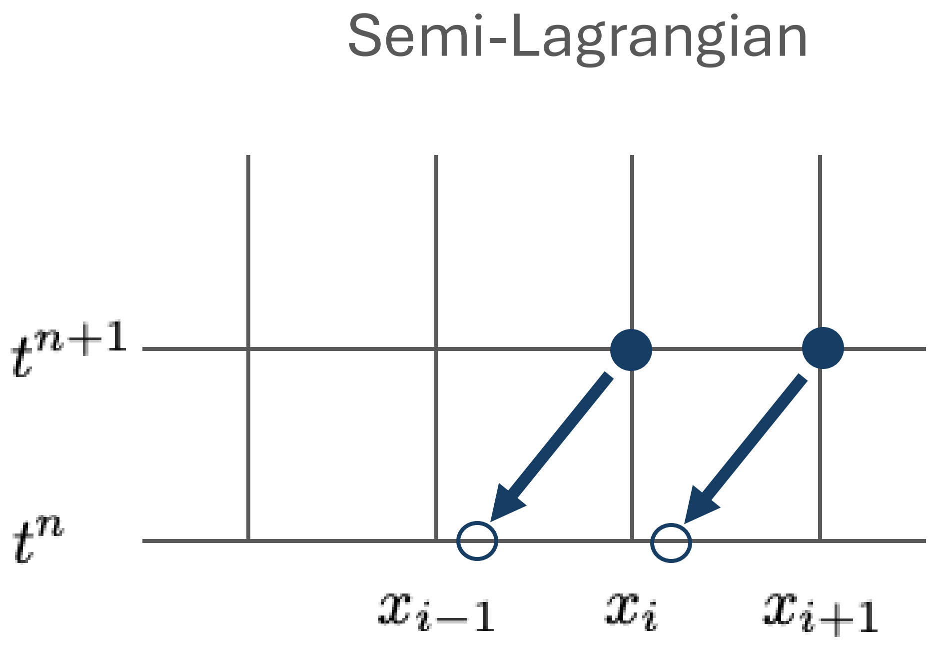
Semi-Lagrangian



Semi-Lagrangian Method

- Pros:
  - Unconditionally stable for linear advection (no CFL restriction).
  - Very good at resolving fronts, sub-grid features.

- Cons:
  - Not strictly conservative.
  - Grid search and interpolation.



Reference Map

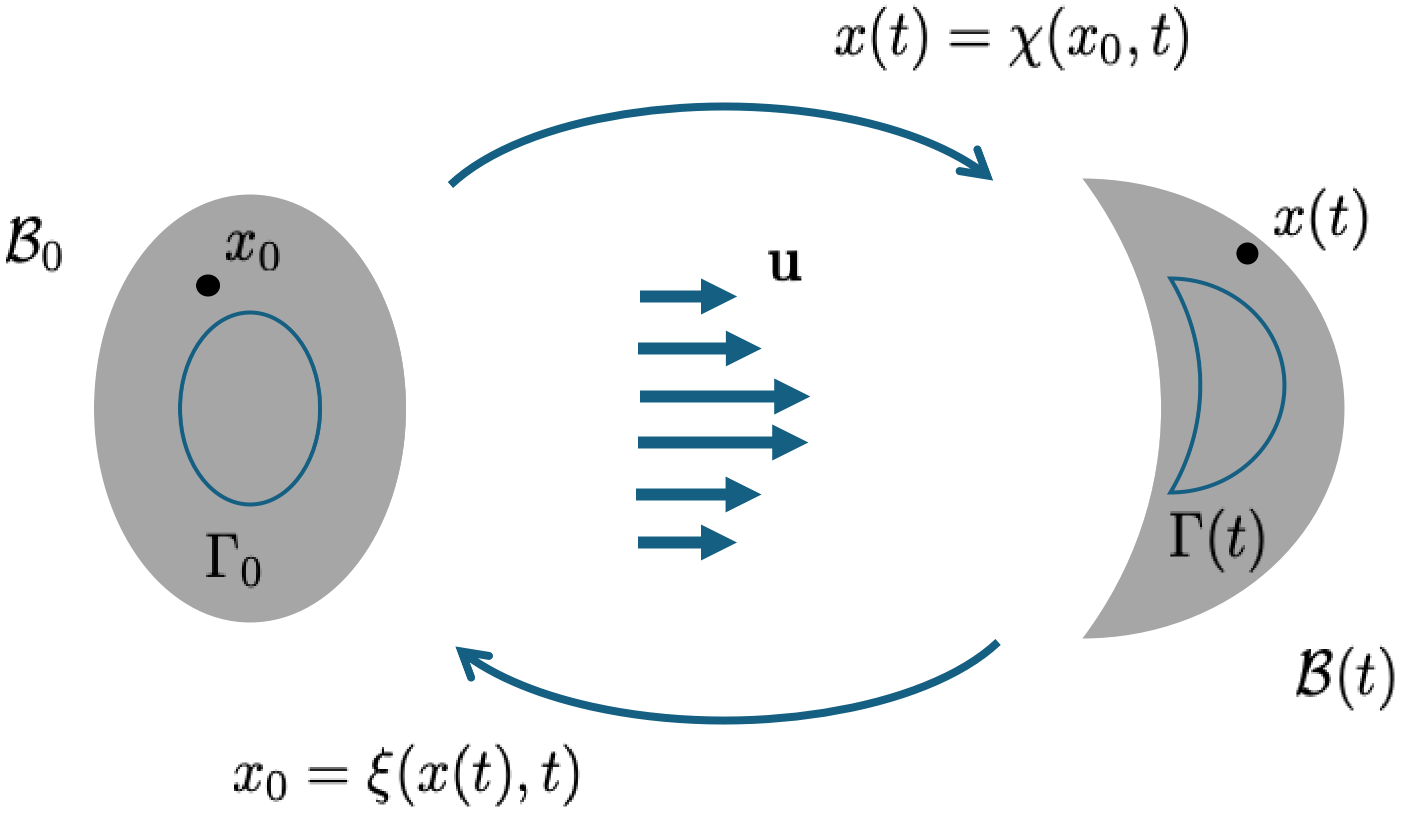
$$\frac{\partial \phi}{\partial t} + \mathbf{u} \cdot \nabla \phi = 0$$

Advection

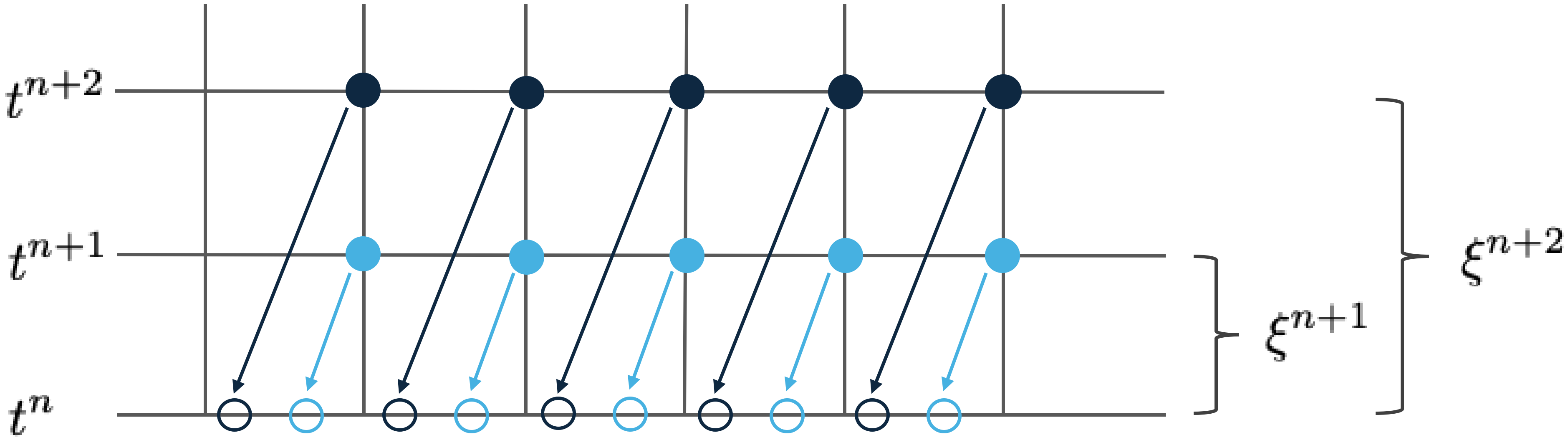
$$\frac{\partial \xi}{\partial t} + \mathbf{u} \cdot \nabla \xi = 0$$

Reconstruction

$$\phi(x, t) = \phi(\xi(x(t), 0)$$

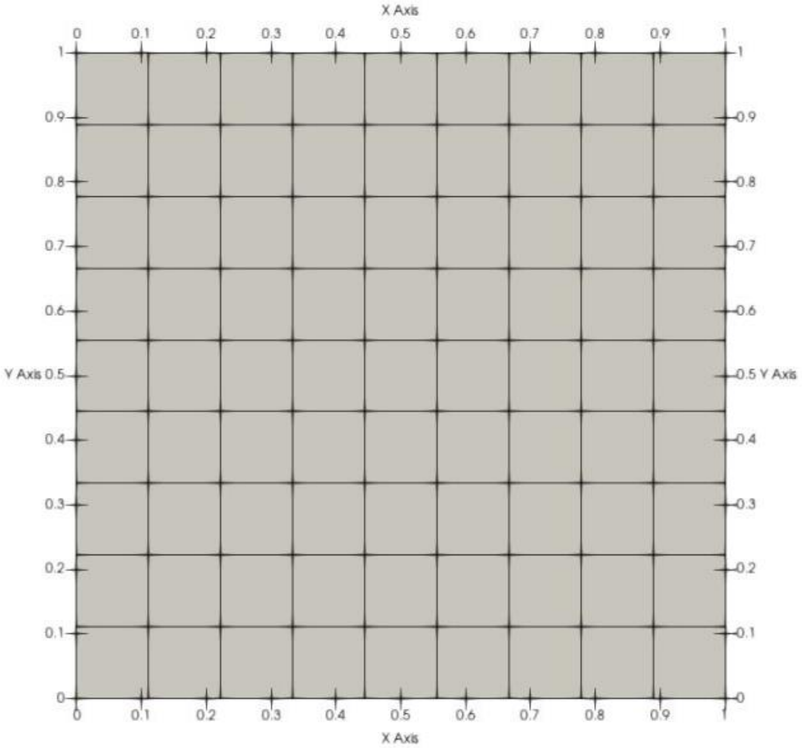


Semi-Lagrangian as the Reference Map



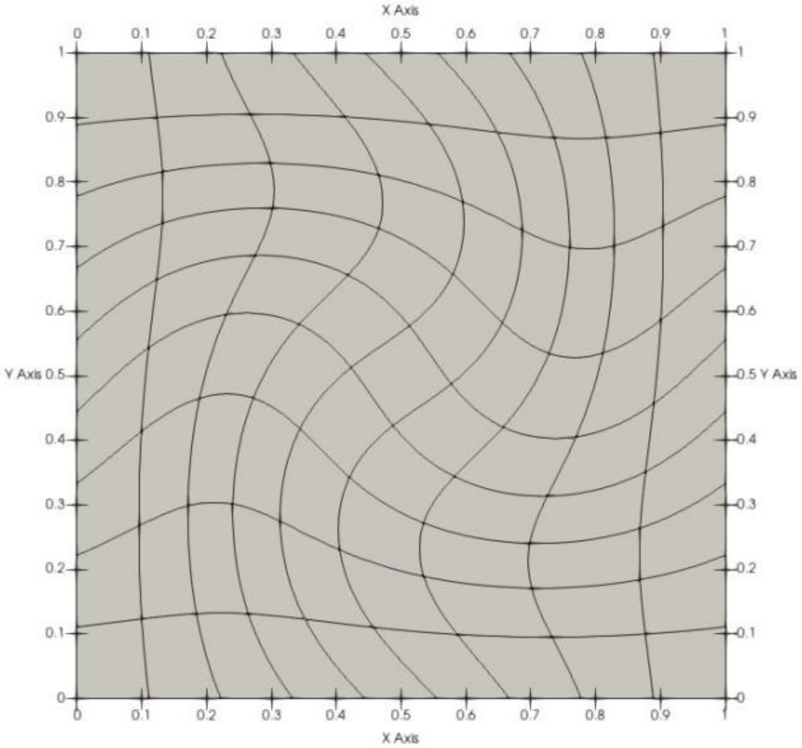
Gradient of the Reference Map

$$\nabla \xi^0$$



$$det(\nabla \xi) = 1$$

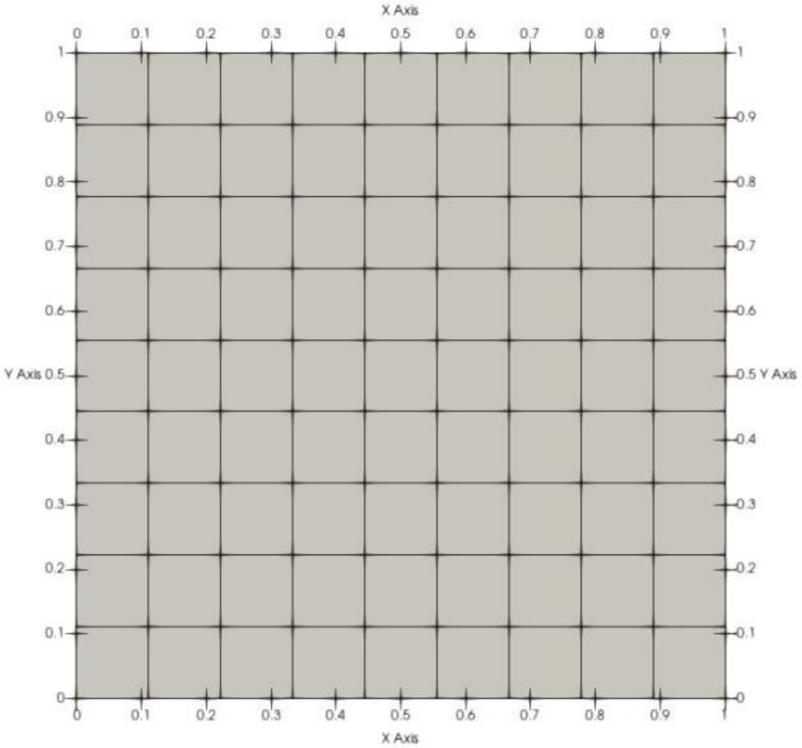
$$\nabla \xi^n$$



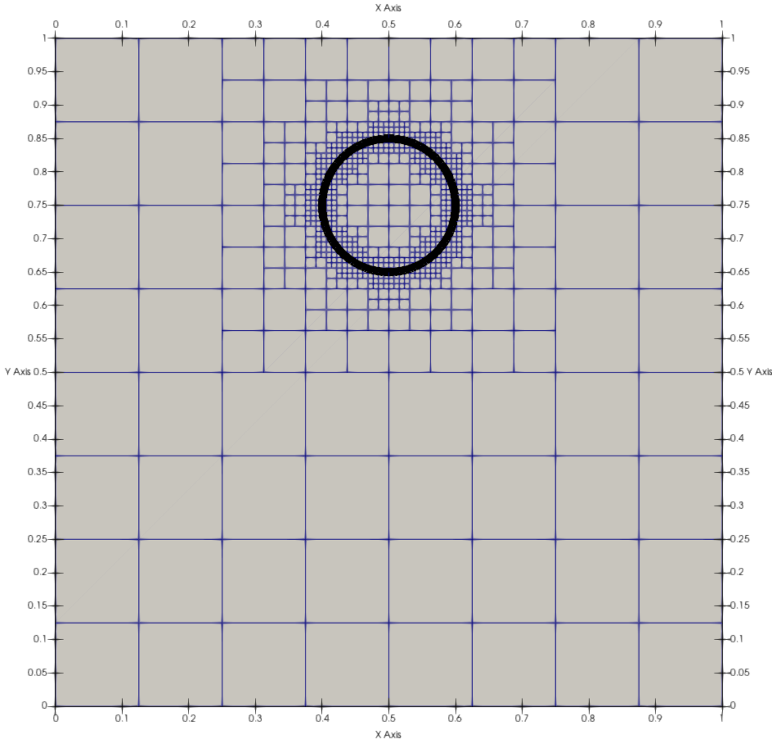
$$det(\nabla \xi) \neq 1$$

Contours of $\nabla\xi$ vs. Adaptive Mesh

$\nabla\xi$



Adaptive Mesh



Volume Preserving Projection

1. Advect the Reference Map

$$\frac{\partial \xi}{\partial t} + \mathbf{u} \cdot \nabla \xi = 0$$

2. Project onto the Volume Preserving Space

Solve for the adjoint λ:

$$-\Delta \lambda = 1 - det(\nabla \xi) \quad \forall x \in \Omega$$

$$\lambda = 0 \quad \forall x \in \partial \Omega$$

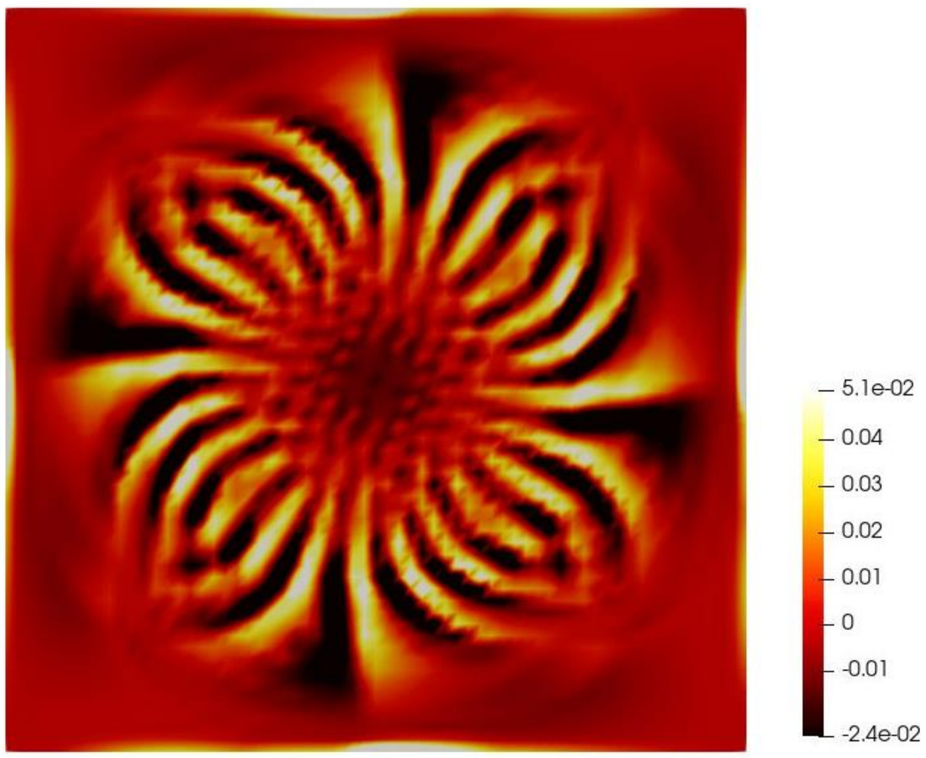
Compute the correction

$$\gamma^{-1}(x) = x - \nabla \lambda$$

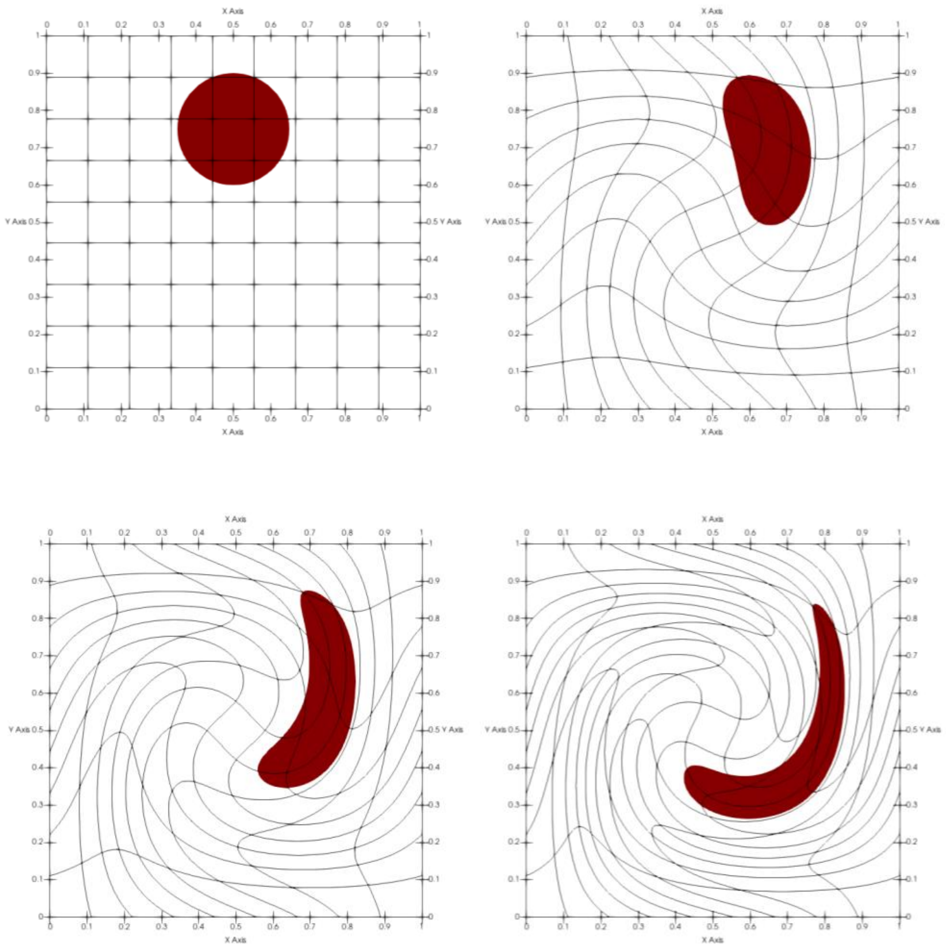
3. Reconstruction

$$\phi^{n+1}(x) = \phi_0 \left(\xi^{n+1} \left(\gamma^{-1} (x) \right) \right)$$

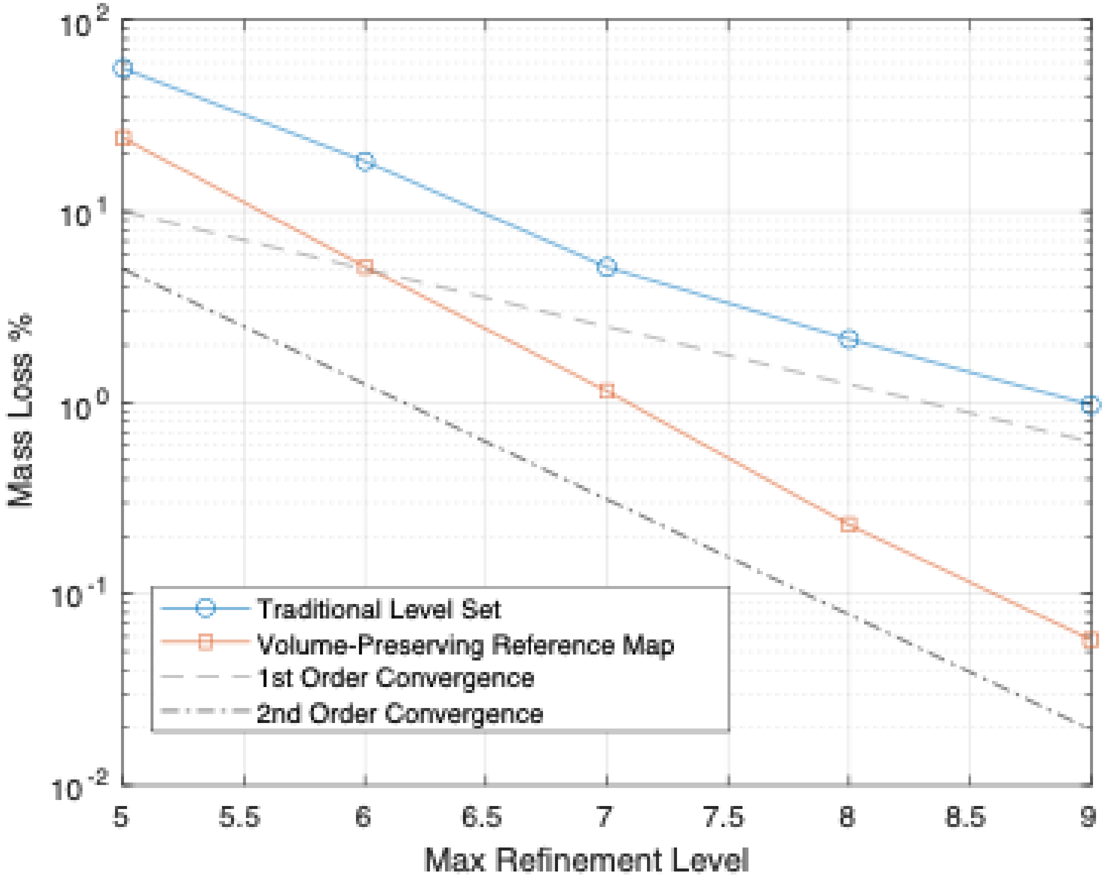
$1 - det(\nabla \xi)$ for the Single Vortex example



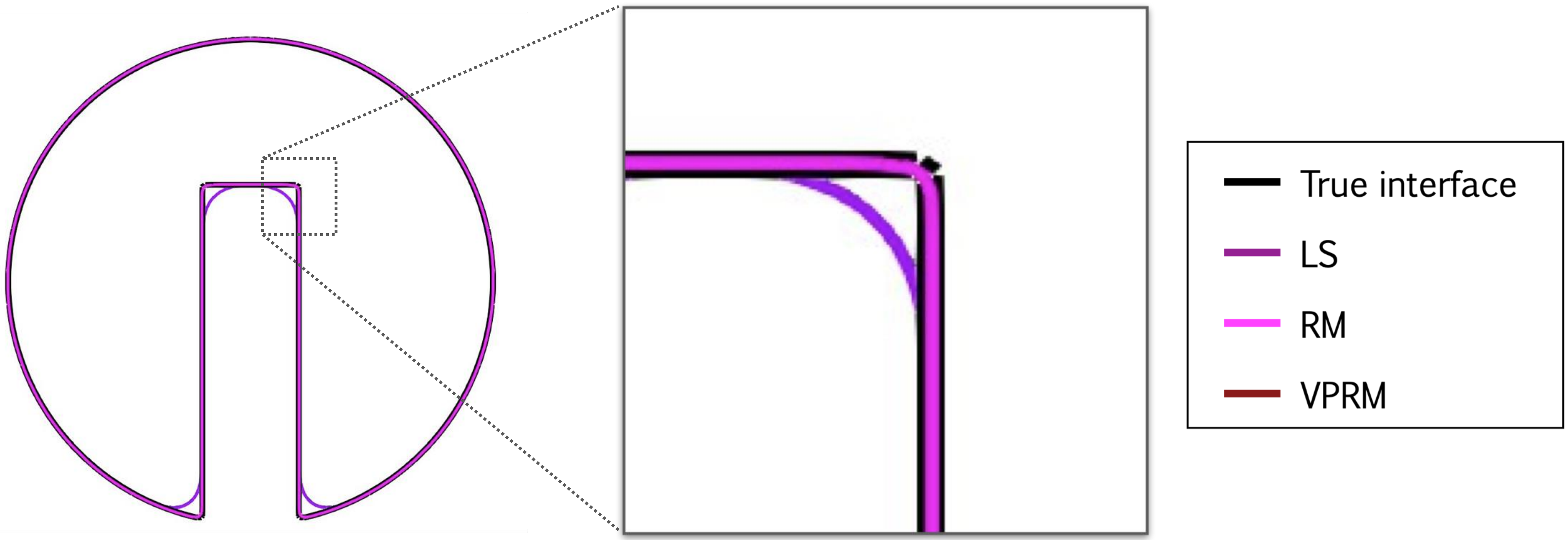
Single Vortex



Mass Loss for Single Vortex



Slotted Circle



Summary

1. Reference map formulation provides a metric for measuring the deformation of the space.
2. Volume preserving projection minimizes mass loss.
3. Together, this preserves interface location and prevents mass loss.

Joint work with:

Adam Binswanger, Maxime Theillard (advisor), Scott West

Papers

A. Binswanger, M. Blomquist, S. R. West, and M. Theillard, "A stable nodal projection method for two-phase flows", In Preparation.

M. Blomquist, S. R. West, A. L. Binswanger, and M. Theillard, "Stable nodal projection method on octree grids", Journal of Computational Physics 499, 112695 (2024).

M. Theillard, A volume-preserving reference map method for the level set representation, Journal of Computational Physics 422, 110478 (2021).

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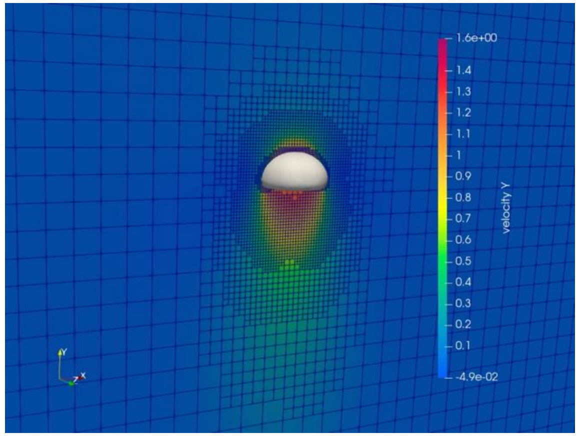
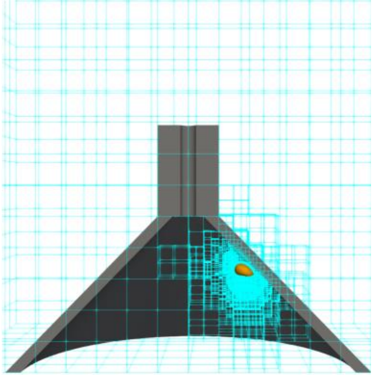
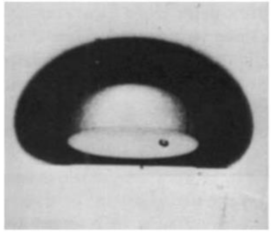




Simulation



Bhaga, Weber '81