Clip and Scale Limiting for Advection-Based Remap of Continuous Fields

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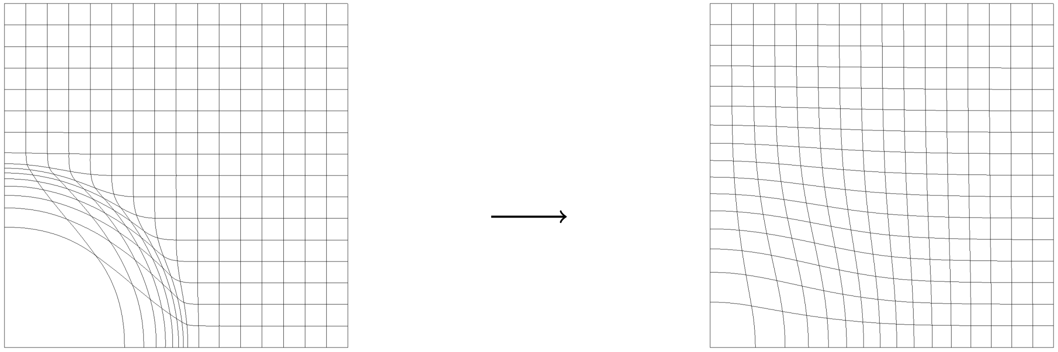
Problem statement

Advection-based remap of a scalar quantity u

$$\frac{\mathrm{d}u}{\mathrm{d}\tau} = \boldsymbol{\nu} \cdot \nabla u, \quad \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}\tau} = \boldsymbol{\nu}, \quad \text{in } \Omega \times [0,1]$$

- Mesh velocity of the i-th node defined by[1]

$$\boldsymbol{\nu}(\mathbf{x}_i) = \mathbf{x}_i\big|_{\tau=1} - \mathbf{x}_i\big|_{\tau=0}$$



Old mesh ($\tau = 0$) $\longrightarrow$ New mesh ($\tau = 1$)

- Boundary nodes only move along the boundary, i.e. $\boldsymbol{\nu} \cdot \mathbf{n} = 0$ on $\partial\Omega$

[1] (Anderson et al., 2015), (Anderson et al., 2018)

Notation

- $\{\varphi_i\}_{i=1}^{N_h}$ is the continuous high-order Bernstein basis on mesh $\mathcal{T}(\tau) = \{K_e(\tau)\}_{e=1}^{E_h}$

- $u_h(\mathbf{x}, \tau) = \sum_{i=1}^{N_h} u_i(\tau)\varphi_i(\mathbf{x}, \tau)$ is a grid function

- $\mathcal{E}_i$ is the index set of elements which share the i-th node

- $\mathcal{N}^e$ is the index set of nodes belonging to the e-th element

- $\mathcal{N}_i$ is the stencil of the i-th node, i.e., all $j \in \{1, \ldots, N\}$ where $\mathrm{supp}(\varphi_i) \cap \mathrm{supp}(\varphi_j) \neq \emptyset$.

- $\mathcal{N}_i^* = \mathcal{N}_i \setminus \{i\}$

- $\mathcal{N}_i^e = \mathcal{N}_i \cap \mathcal{N}^e$ and $\mathcal{N}_i^{e*} = \mathcal{N}_i^* \cap \mathcal{N}^e$

CG DISCRETIZATION

- Owing to the Reynolds Transport theorem, remap can be written as

$$M \frac{\mathrm{d}u}{\mathrm{d}\tau} = Ku$$

with matrix entries

$$m_{ij} = \sum_{e \in \mathcal{E}_i} \underbrace{\int_{K_e(\tau)} \varphi_i \varphi_j}_{m_{ij}^e}, \quad k_{ij} = \sum_{e \in \mathcal{E}_i} \underbrace{\int_{K_e(\tau)} \varphi_i (\boldsymbol{\nu} \cdot \nabla \varphi_j)}_{k_{ij}^e}$$

- Mass lumping and addition of element-based artificial diffusion yields low-order (LO) scheme[2]

$$m_i \dot{u}_i = \sum_{e \in \mathcal{E}_i} \sum_{j \in \mathcal{N}_i^{e*}} (d_{ij}^e + k_{ij}^e)(u_j - u_i),$$

where

$$m_i = \sum_{j \in \mathcal{N}_i} m_{ij} > 0$$

$$d_{ij}^e = \begin{cases} \max(-k_{ij}^e, -k_{ji}^e, 0) & \text{if } i \in \mathcal{N}^e \text{ and } j \in \mathcal{N}_i^{e*}, \\ -\sum_{k \in \mathcal{N}_i^{e*}} d_{ik}^e & \text{if } i \in \mathcal{N}^e \text{ and } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

[2](Kuzmin and Hajduk, 2023)

Bound-preservation of LO scheme

- Bar-state form of Semi-discrete LO method

$$m_i \dot{u}_i = \sum_{e \in \mathcal{E}_i} \sum_{j \in \mathcal{N}_i^{e*}} (d_{ij}^e + k_{ij}^e)(u_j - u_i) = \sum_{e \in \mathcal{E}_i} 2 d_{i,e} (\overline{u}_{i,e} - u_i)$$

- (Pseudo-) time discretization with explicit SSP-RK method

$$\begin{aligned} u_i^{\mathrm{SSP}} &= u_i + \frac{\Delta\tau}{m_i} \sum_{e \in \mathcal{E}_i} 2 d_{i,e} (\overline{u}_{i,e} - u_i) \\ &= \left(1 - \frac{\Delta\tau}{m_i} \sum_{e \in \mathcal{E}_i} 2 d_{i,e}\right) u_i + \frac{\Delta\tau}{m_i} \sum_{e \in \mathcal{E}_i} 2 d_{i,e} \overline{u}_{i,e} \end{aligned}$$

- u_i^{SSP} is convex combination of u_i and $\overline{u}_{i,e}$, $e \in \mathcal{E}_i$ under time step restiction

$$\frac{\Delta\tau}{m_i} \sum_{e \in \mathcal{E}_i} 2 d_{i,e} \leq 1$$

- Therefore,

$$u_i^{\min} := \min_{j \in \mathcal{N}_i} u_j \leq u_i^{\mathrm{SSP}} \leq \max_{j \in \mathcal{N}_i} u_j := u_i^{\max}$$

Antidiffusive fluxes

- Recover high-order scheme by addition of

$$f_{ij}^e = m_{ij}^e(\dot{u}_i^L - \dot{u}_j^L) + d_{ij}^e(u_i - u_j), \quad f_{i,e} = \sum_{j \in \mathcal{N}_i^{e*}} f_{ij}^e$$

  to obtain

$$m_i \dot{u}_i = \sum_{e \in \mathcal{E}_i} 2d_{i,e}(\overline{u}_{i,e} - u_i) + f_{i,e}$$

- Replace $f_{i,e}$ by limited counter part $f_{i,e}^*$ and include in bar-states $\overline{u}_{i,e}^* := \overline{u}_{i,e} + \frac{f_{i,e}^*}{2d_{i,e}}$

$$m_i \dot{u}_i = \sum_{e \in \mathcal{E}_i} 2d_{i,e}(\overline{u}_{i,e}^* - u_i)$$

- Limit such that

$$u_i^{\min} \leq \overline{u}_{i,e}^* \leq u_i^{\max}$$
$$\Leftrightarrow$$
$$f_{i,e}^{\min} := 2d_{i,e}(u_i^{\min} - \overline{u}_{i,e}) \leq f_{i,e}^* \leq 2d_{i,e}(u_i^{\max} - \overline{u}_{i,e}) := f_{i,e}^{\max}.$$

  and therefore, the convexity argument holds

Clip and scale limiting

Clip

$$\tilde{f}_{i,e} = \max(\min(f_{i,e}^{\max}, f_{i,e}), f_{i,e}^{\min})$$

- For mass conservation we need $\sum_{i \in \mathcal{N}^e} f_{i,e}^* = 0$
- Define

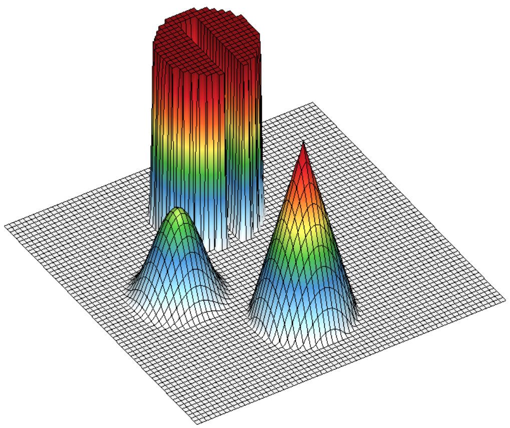
$$P_e^+ = \sum_{i \in \mathcal{N}^e} \max(0, \tilde{f}_{i,e}), \quad P_e^- = \sum_{i \in \mathcal{N}^e} \min(0, \tilde{f}_{i,e})$$

and Scale

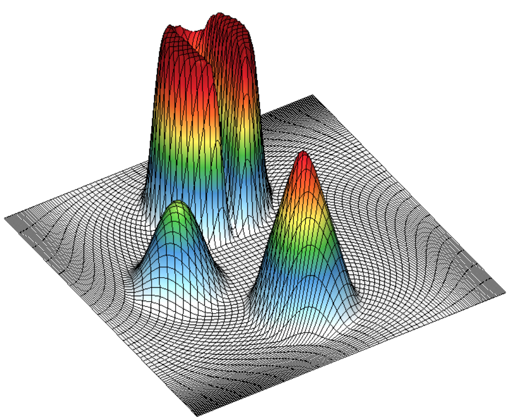
$$f_{i,e}^* = \begin{cases} -\frac{P_e^-}{P_e^+} \tilde{f}_{i,e} & \text{if } \tilde{f}_{i,e} > 0 \text{ and } P_e^+ + P_e^- > 0 \\ -\frac{P_e^+}{P_e^-} \tilde{f}_{i,e} & \text{if } \tilde{f}_{i,e} < 0 \text{ and } P_e^+ + P_e^- < 0 \\ \tilde{f}_{i,e} & \text{otherwise.} \end{cases}$$

Numerical examples

- Solid Body "Rotation" ($\mathbb{Q}_2$, SSP3-RK, mesh order 3)

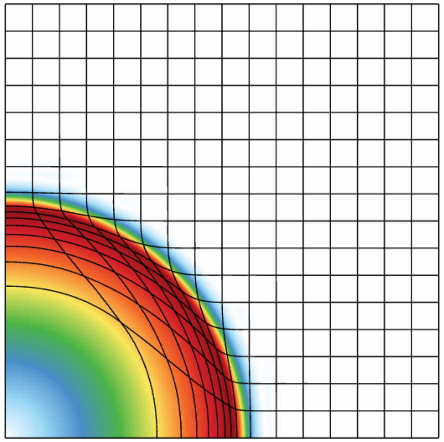


Initial condition on uniform mesh, $u \in [0,1]$

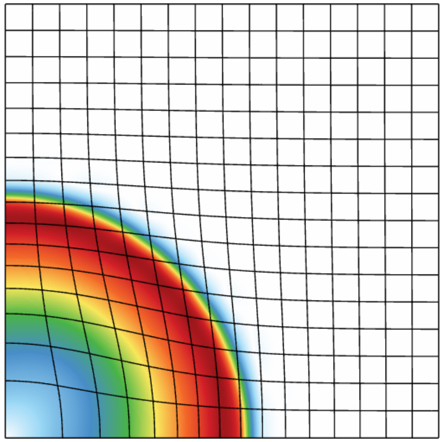


Remapped solution, $u \in [0,1]$

- 2D Sedov Blast problem ($\mathbb{Q}_2$, SSP3-RK, mesh order 2)



Pre remap velocity, $|\mathbf{v}| \in [0, 0.7704]$



Remapped velocity, $|\mathbf{v}| \in [0, 0.7372]$

Conclusions

Clip and Scale Limiter applied to advection-based remap

- stable and nonoscillatory solutions
- Shock capturing and bound preservation
- shapes of initial condition preserved

Work in Progress

- Implementation for linear advection in *ex9* and *ex9p* (pull request #4505)
- Boundary penalty term to enforce $\mathbf{v} \cdot \mathbf{n} = 0$ for velocity remap
- Subcell Clip and Scale Limiting

References

- R. Anderson, V. Dobrev, T. Kolev, R. Rieben, Monotonicity in high-order curvilinear finite element arbitrary Lagrangian–Eulerian remap. *Internat. J. Numer. Methods Fluids* **77** (2015), 249-273.
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- D. Kuzmin and H. Hajduk, Property-Preserving Numerical Schemes for Conservation Laws. *World Scientific* (2023)