Robust Containment Queries over Collections of Parametric Curves via Generalized Winding Numbers

Jacob Spainhour[1, 2], Kenneth Weiss[2]

[1]Department of Applied Mathematics, University of Colorado Boulder

[2]Lawrence Livermore National Laboratory

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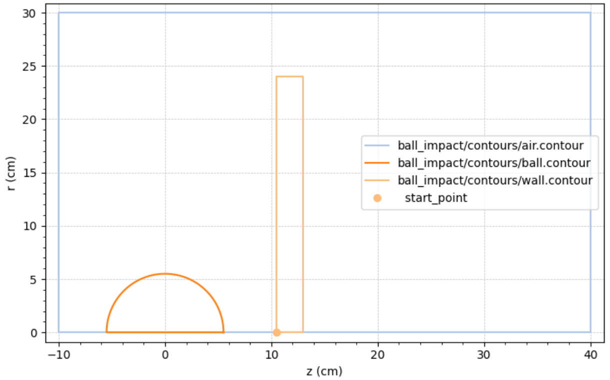
LLNL-PRES-870858

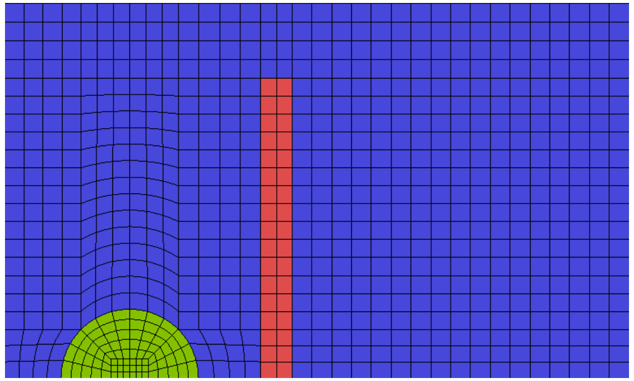
Motivation: Shaping for Multimaterial Simulations

Initializing simulation data like volume fractions and moments from geometric objects is critically important

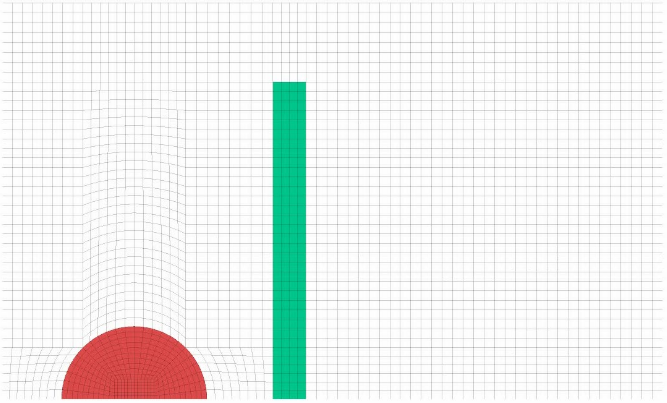
User provided contours



Our meshing tool (PMesh) creates conformal meshes for 2D/3D NURBS



MARBL executes the physics



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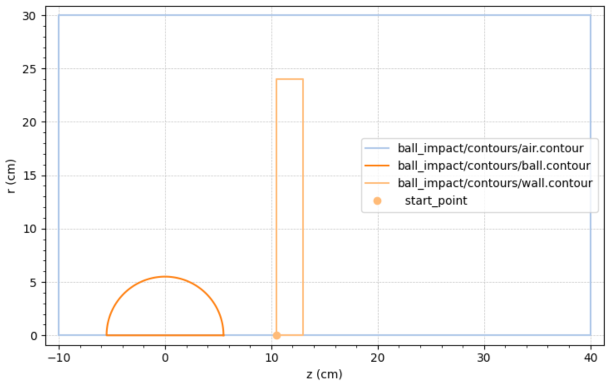
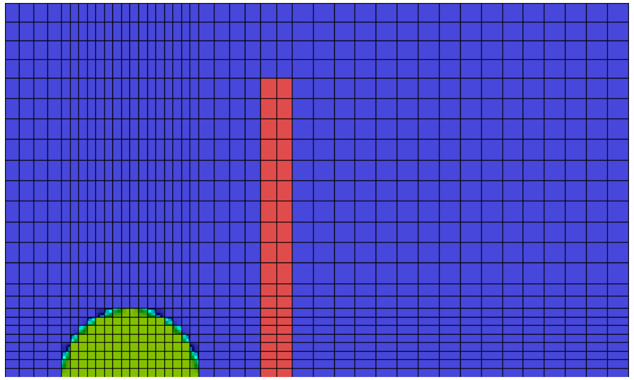


Ball and wall shaped into a logically **Cartesian** mesh



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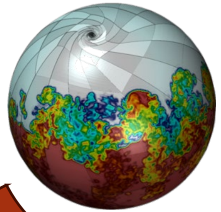
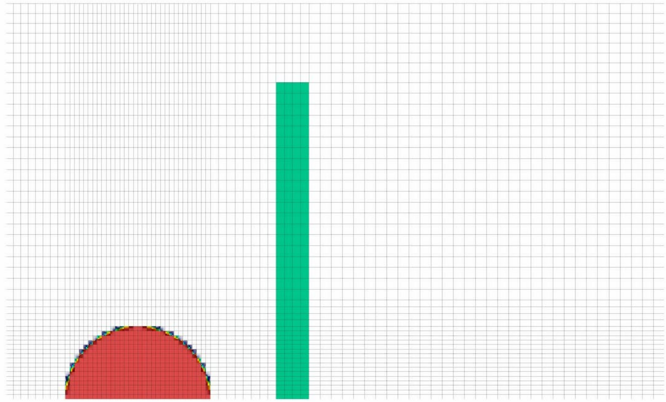
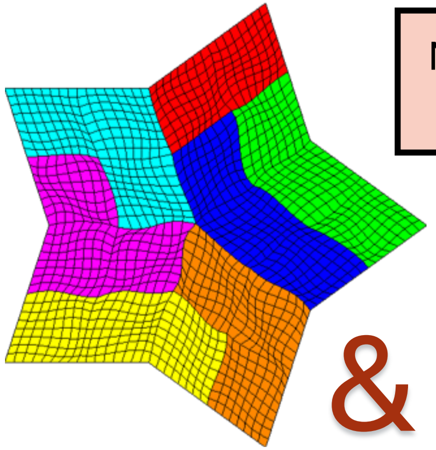




Shaping is required when there is no conformal mesh!

Motivation: Shaping for Multimaterial Simulations

Axom's "Klee" component simplifies shaping through user provided replacement rules



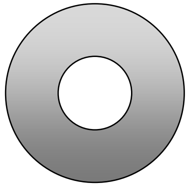
NURBS Mesh

&

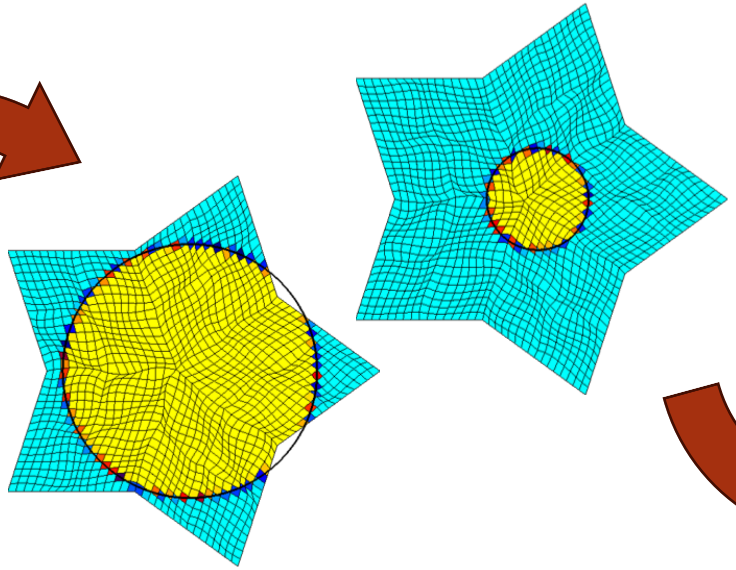
Geometric Objects & Rules

```
dimensions: 2
units: cm
shapes:
  - name: outer_shell
    material: steel
    geometry:
      format: c2c
      path: unit_circle.contour
      units: cm
      operators:
      - scale: 5

  - name: inner_shell
    material: void
    geometry:
```

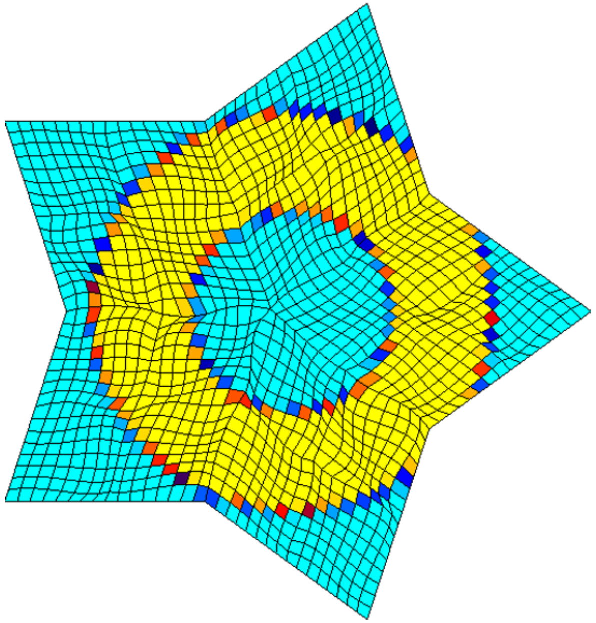


AXOM : Klee



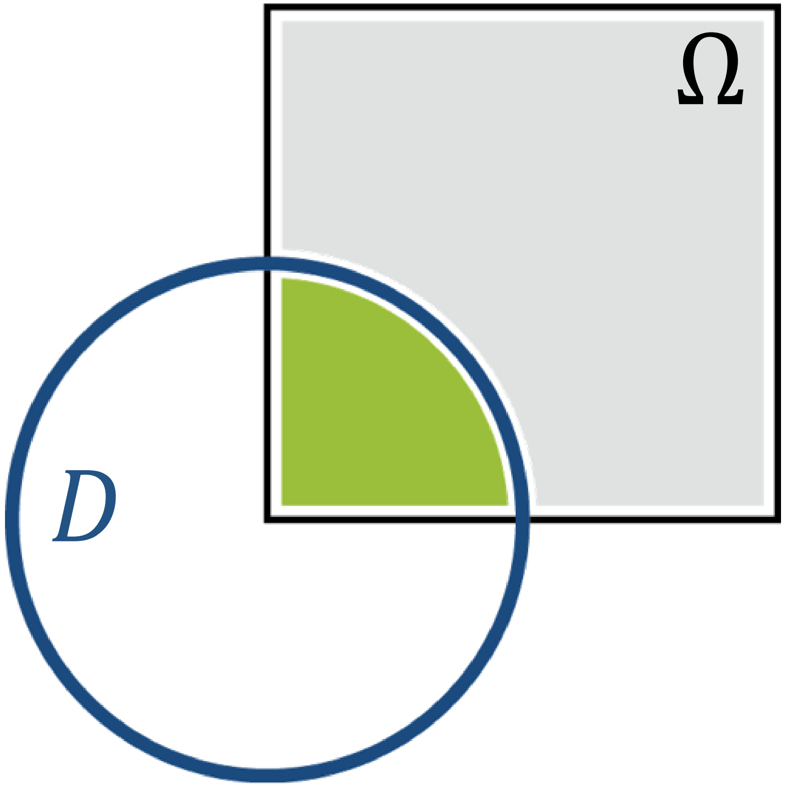
Process individual shapes on each cell

Combine data according to replacement rules



Shaping: Geometric Overlay vs Sampling

Per-cell objective: Compute the volume of the intersection between D and Ω

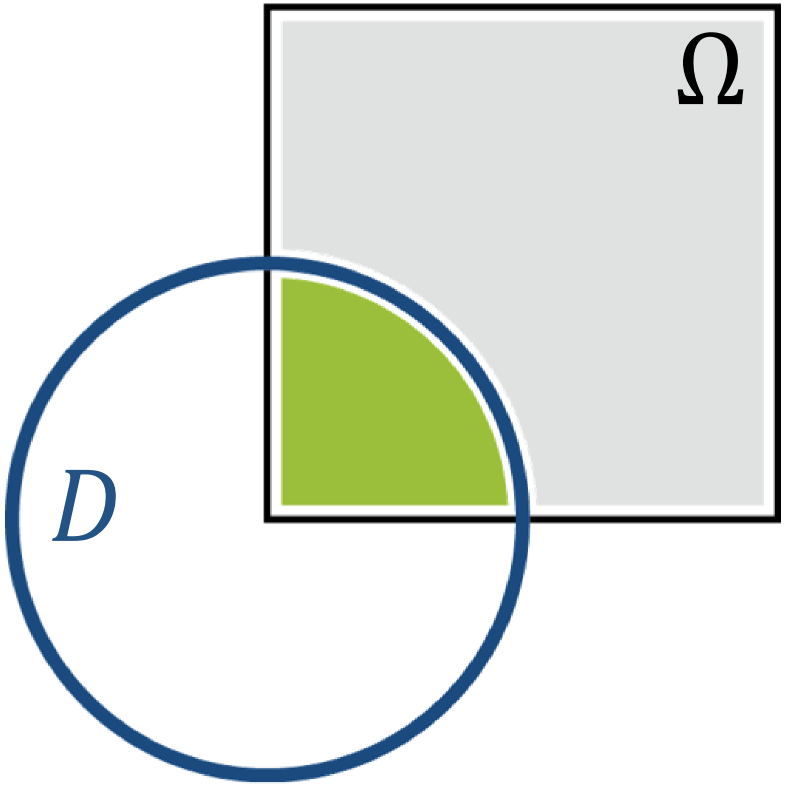


$$f = \iint_{\Omega \cap D} dx$$

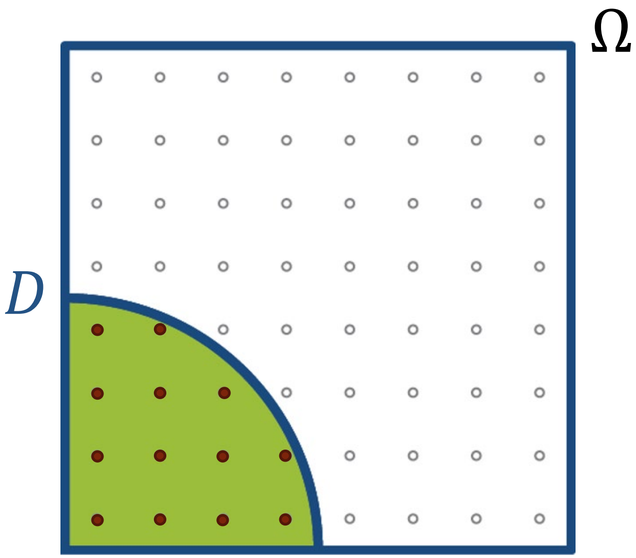
Computing the overlap analytically is difficult even for simple shapes

Shaping: Geometric Overlay vs Sampling

Per-cell objective: Compute the volume of the intersection between D and Ω

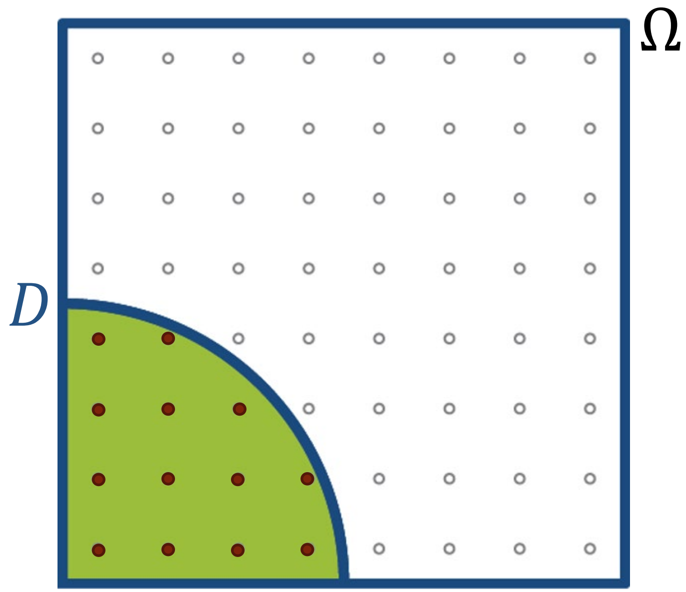


$$f = \iint_{\Omega} I_D(x)\,dx$$

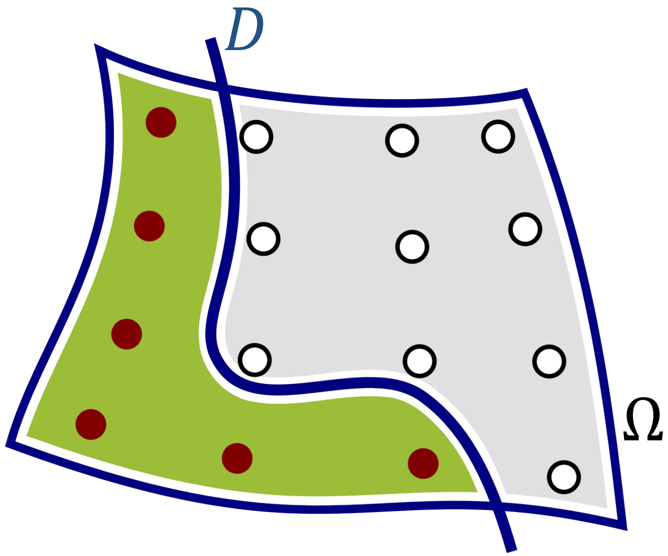


Computing the overlap with quadrature over sampled points is more flexible!

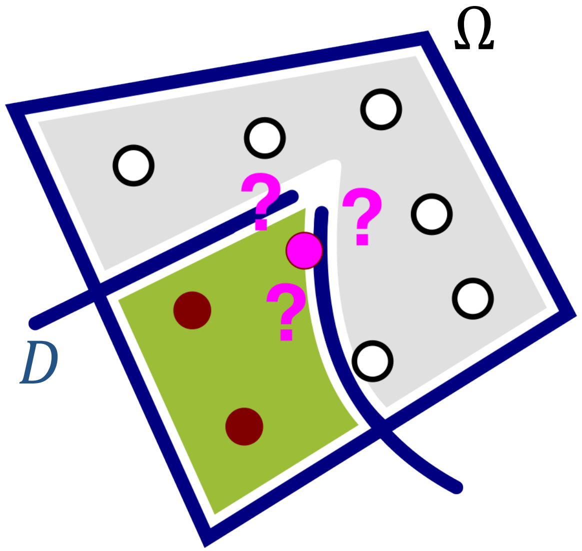
Shaping via Sampling



Circles & Squares ☺



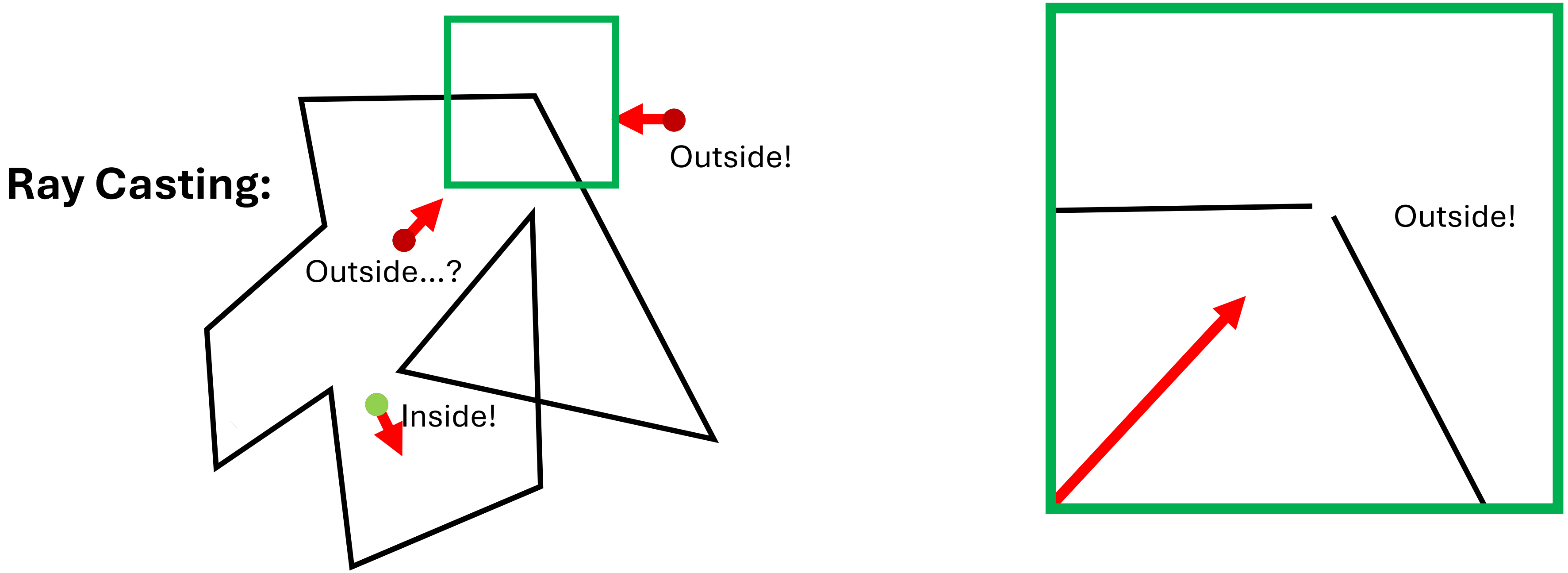
NURBS & Curves 😐



Broken Shapes ☹

Gaps in the boundary make existing containment methods inaccurate or unusable!

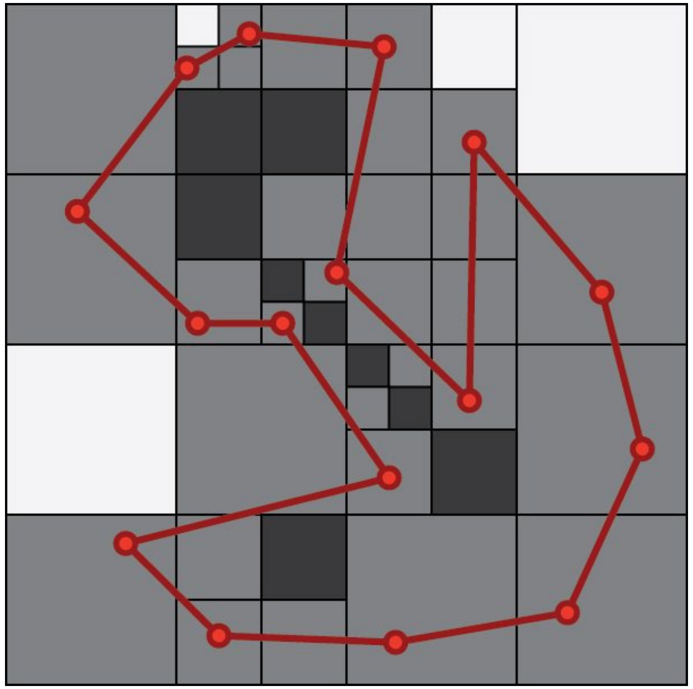
Containment Query Primer: Ray Casting



- Containment is sensitive to watertightness, but in-the-wild shapes are prone to gaps and overlaps

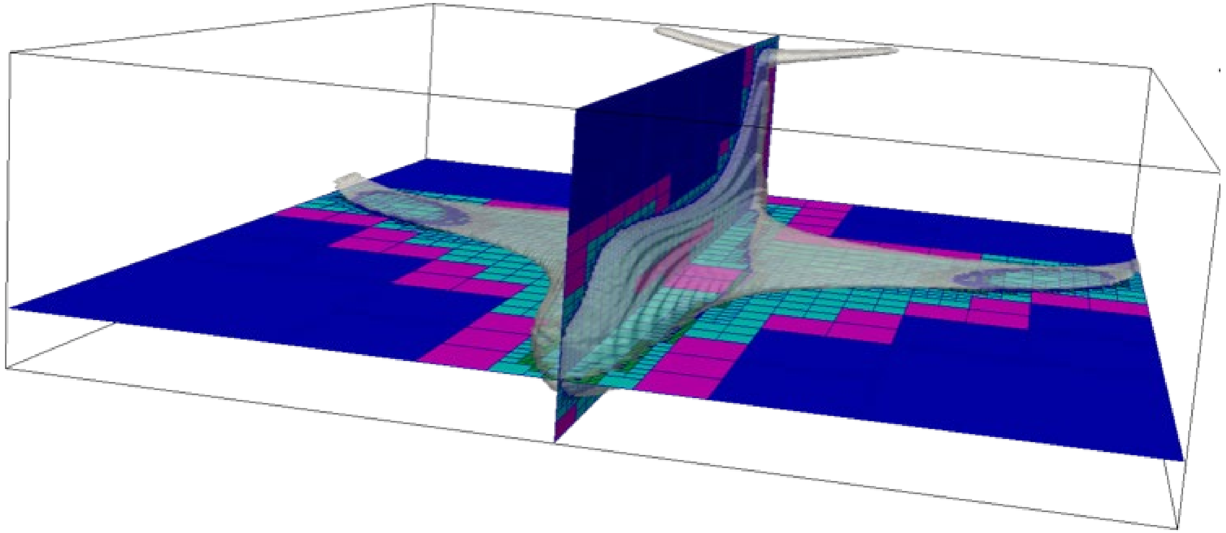
Prior Containment Queries in AXOM: Quest

Linearizes high-order curves



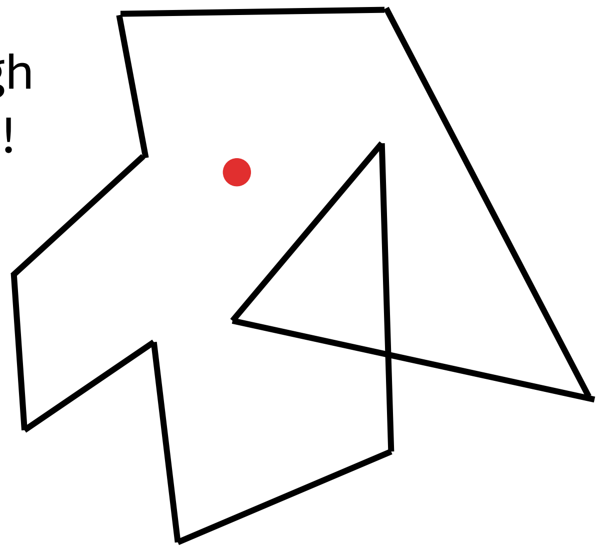
Susceptible to misaligned vertices!

Accelerated using octree data structures



Containment Query Primer: Winding Numbers

Close enough
to be inside!

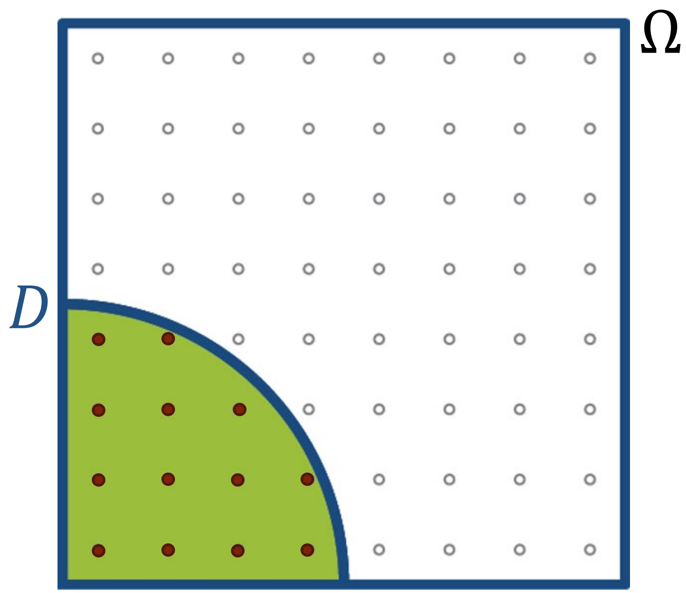


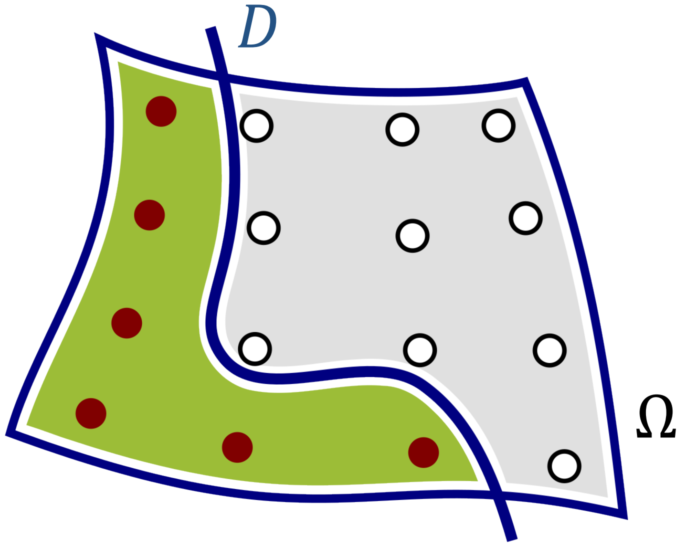
Partial Revolutions per Edge /
Generalized Winding Number:

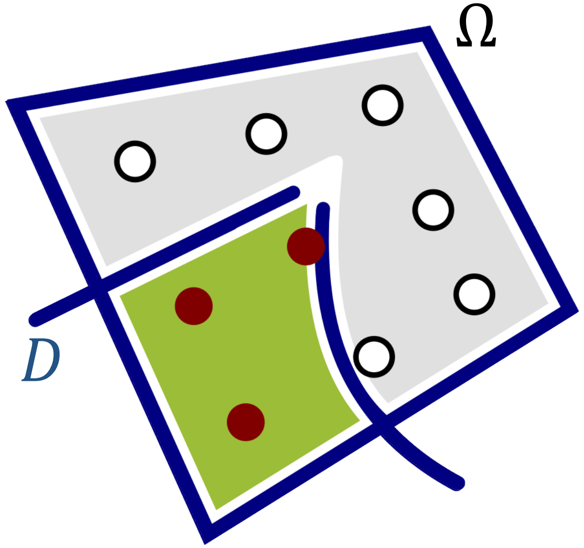
```
  0.2214
+ 0.2200
+ 0.1527
- 0.2651
+ 0.1933
+ 0.1056
+ 0.0462
+ 0.0395
+ 0.0709
+ 0.0890
+ 0.1252
  0.9989
```
≈ 1

- More robust than ray-casting even on watertight geometry!

Shaping via Sampling + Generalized Winding Numbers







$$f = \iint_{\Omega} I_D(x)\,dx = \iint_{\Omega} \mathrm{round}(\mathrm{GWN}_D(x))\,dx$$

Can replace indicator function with the Generalized Winding Number (GWN).

Direct Formula for Generalized Winding Numbers

$$w = \frac{1}{2\pi}\int d\theta$$

Angle Element

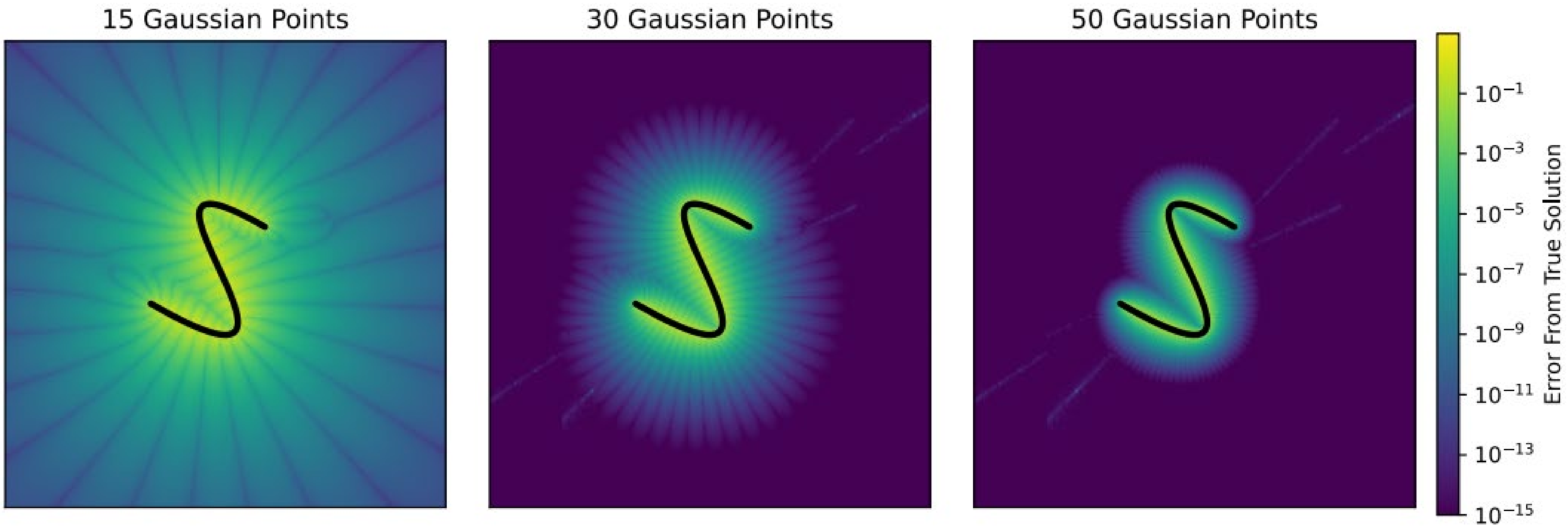
2D

3D

$$w = \frac{1}{4\pi}\iint d\Omega$$

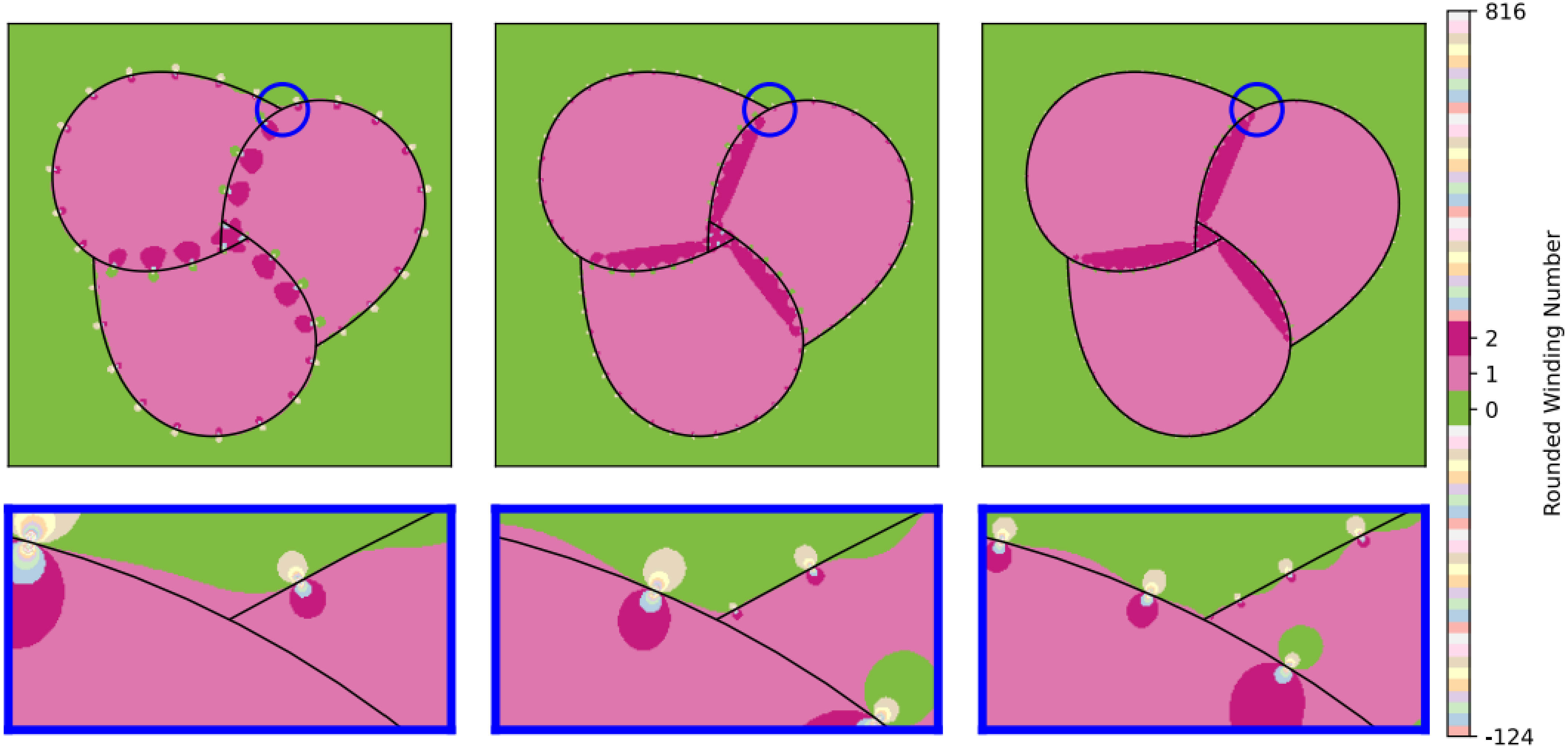
Solid Angle Element

Generalized Winding Numbers: Difficult to Evaluate



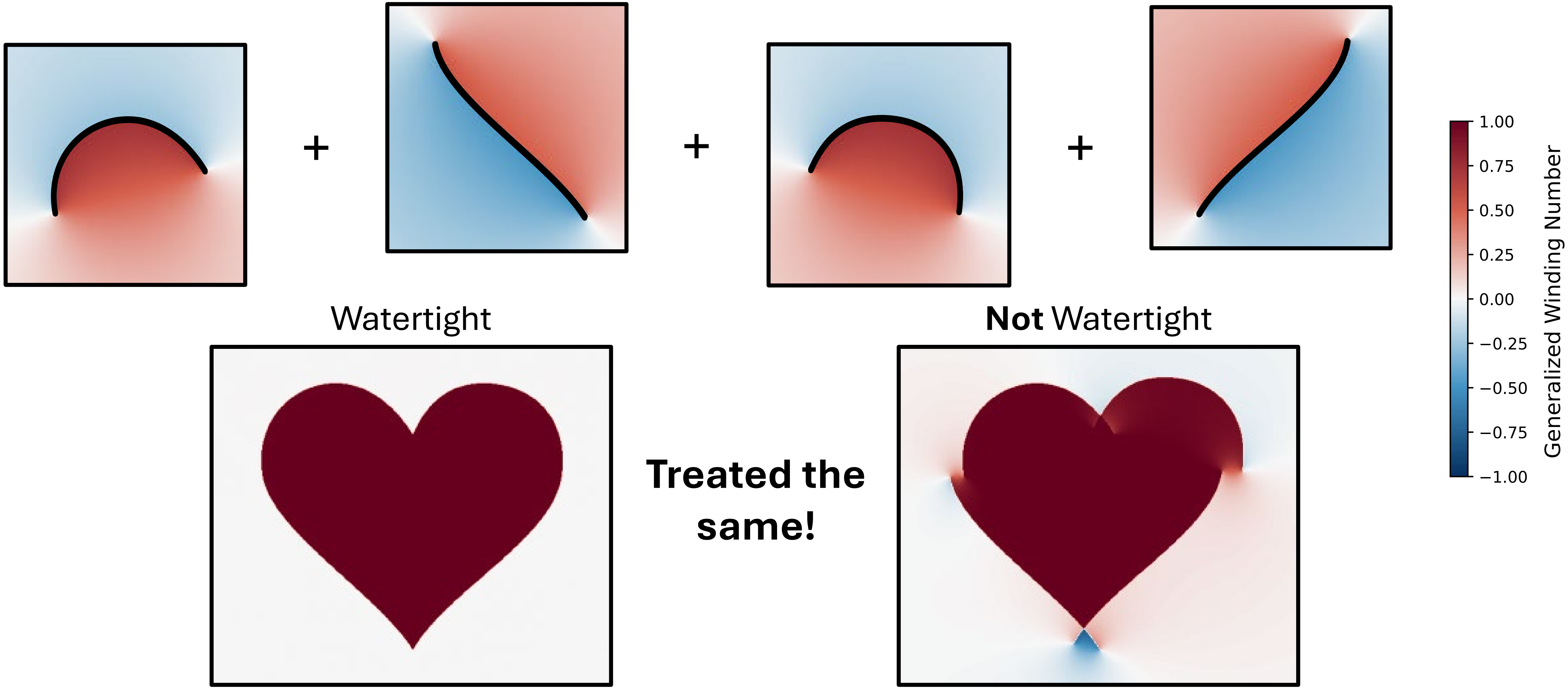
Singularities in the integrand make direct evaluation via quadrature unstable

Generalized Winding Numbers: Difficult to Evaluate



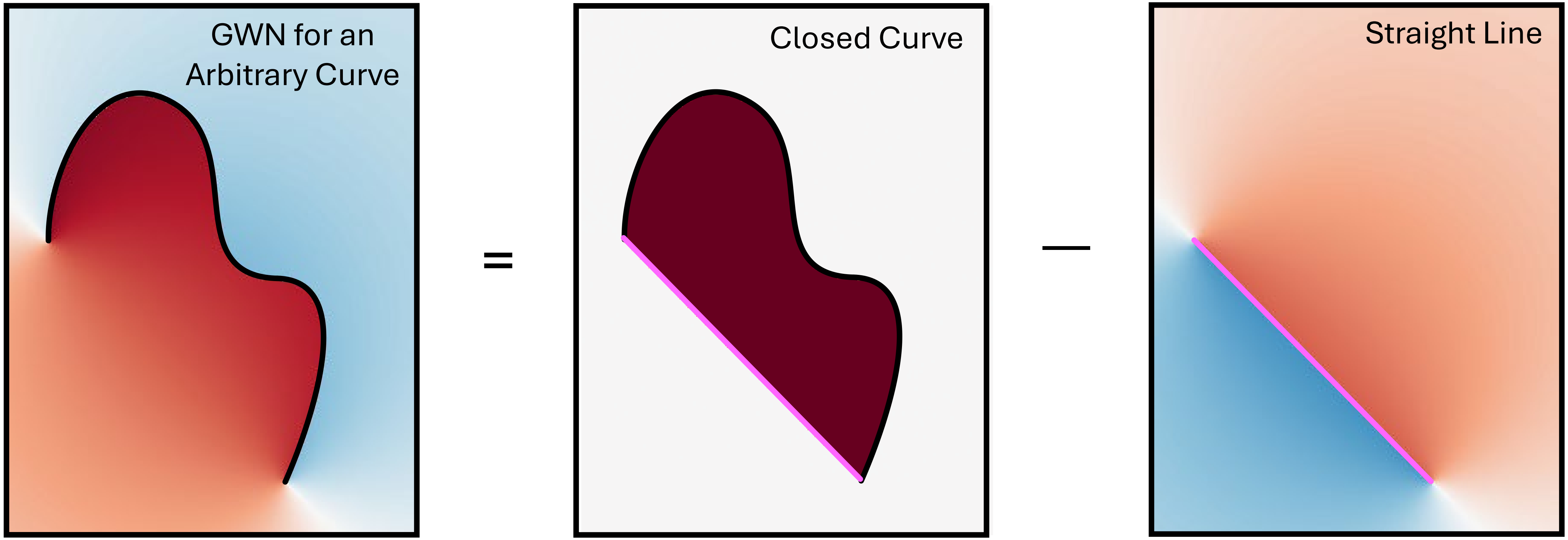
Singularities in the integrand make direct evaluation via quadrature unstable

Generalized Winding Numbers: Indifferent to Watertightness



We compute the GWN from exact, known quantities

Can **both** be computed exactly!



Jacob Spainhour, David Gunderman, and Kenneth Weiss. 2024. **Robust Containment Queries over Collections of Rational Parametric Curves via Generalized Winding Numbers.** *ACM Transactions on Graphics (SIGGRAPH)*

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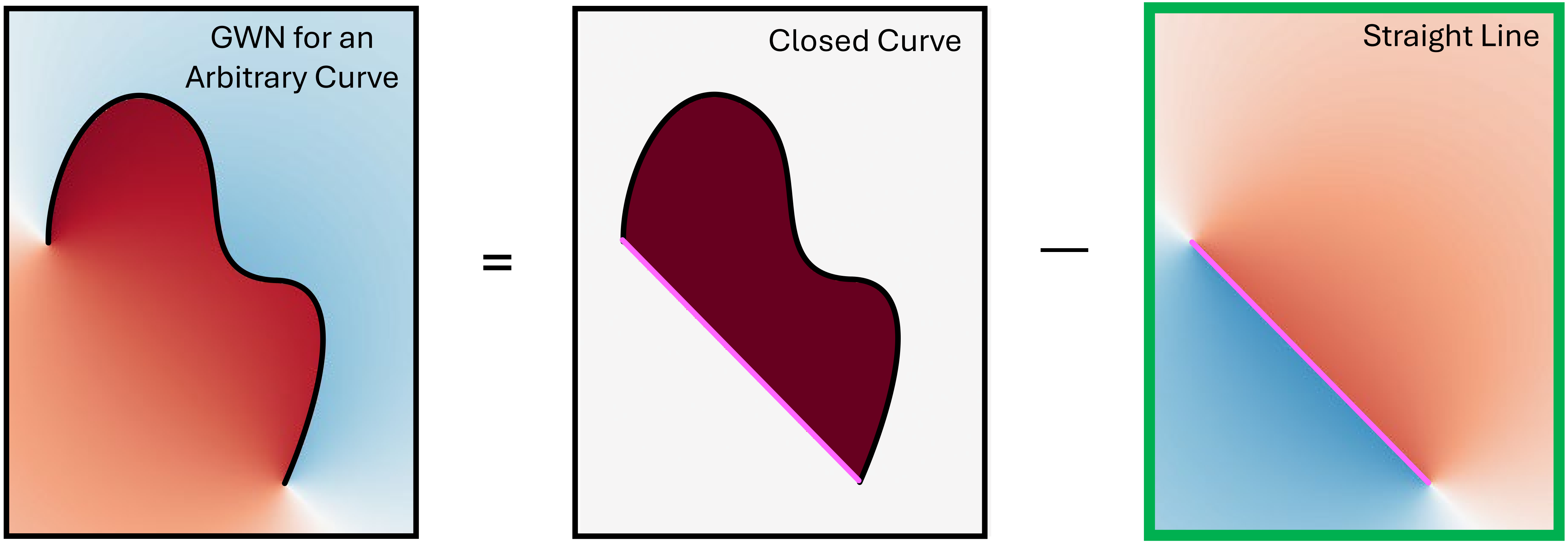

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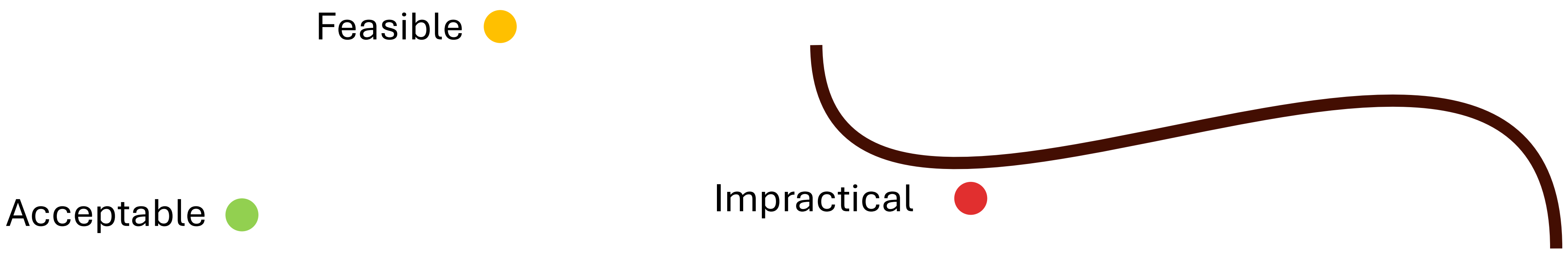
Calculating the GWN for Curves?

- The direct formula is generally unstable, but useful in some cases

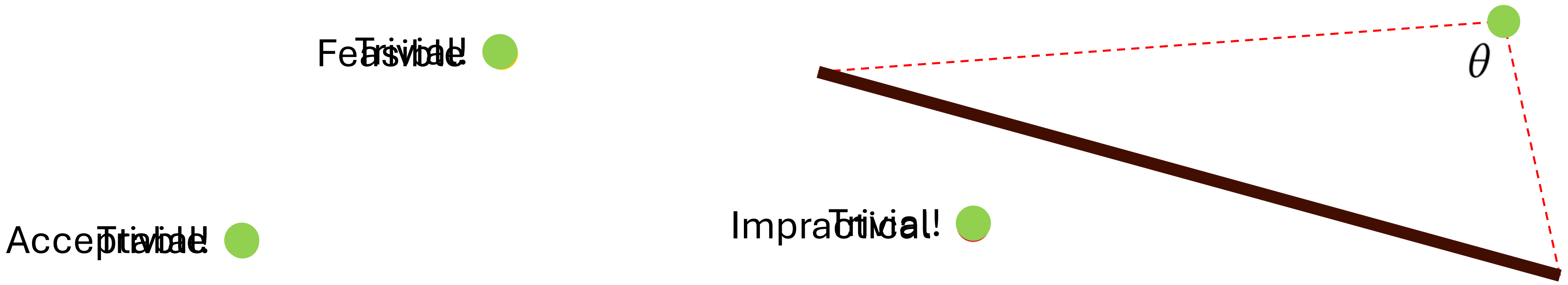
$$\boxed{w_\Gamma(q) := \frac{1}{2\pi}\int_0^1 \frac{x(t)y'(t) - x'(t)y(t)}{x^2(t) + y^2(t)}\,dt}$$



Calculating the GWN for Lines!

- The direct formula is generally unstable, but useful in some cases

$$\boxed{w_\Gamma(q) := \frac{1}{2\pi}\int_0^1 \frac{x(t)y'(t) - x'(t)y(t)}{x^2(t) + y^2(t)}\,dt} \longrightarrow \boxed{w_L(q) = \frac{1}{2\pi}\theta}$$



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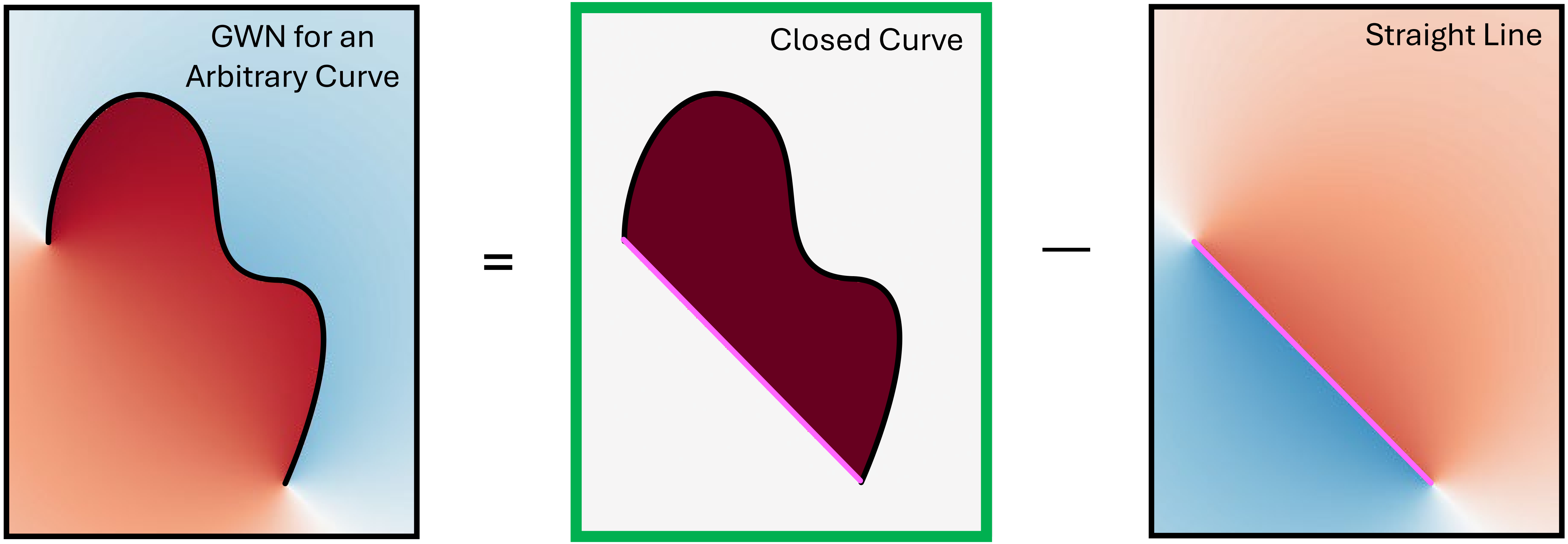

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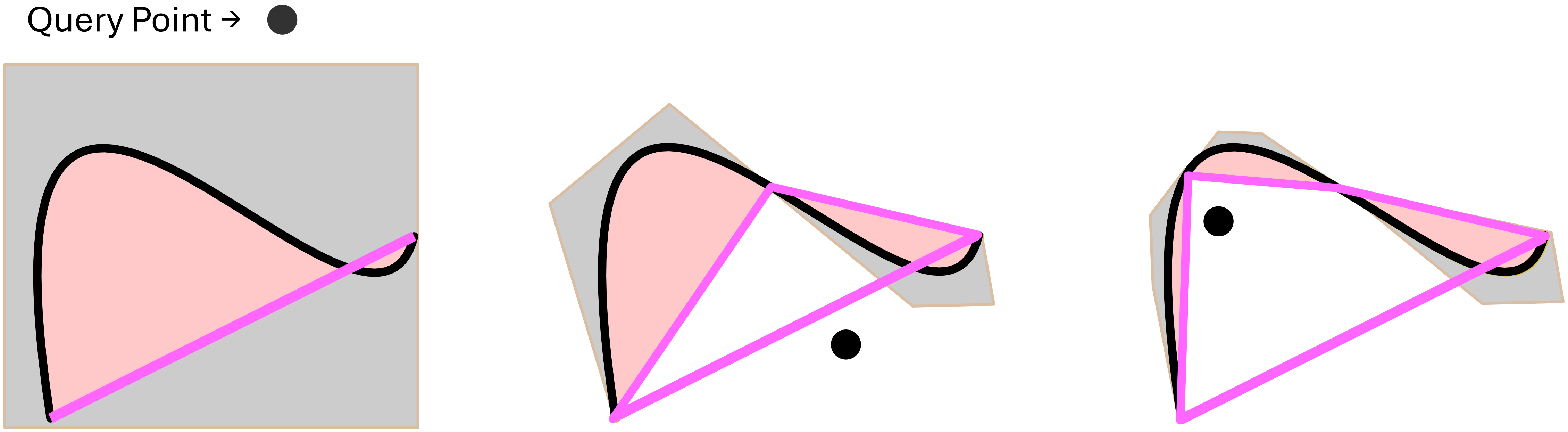
Our Adaptive Integer WN Algorithm Ensures Exactness

Recursively subdivide until the integer winding number is provably zero for each closed subcurve



Outside Bounding Box

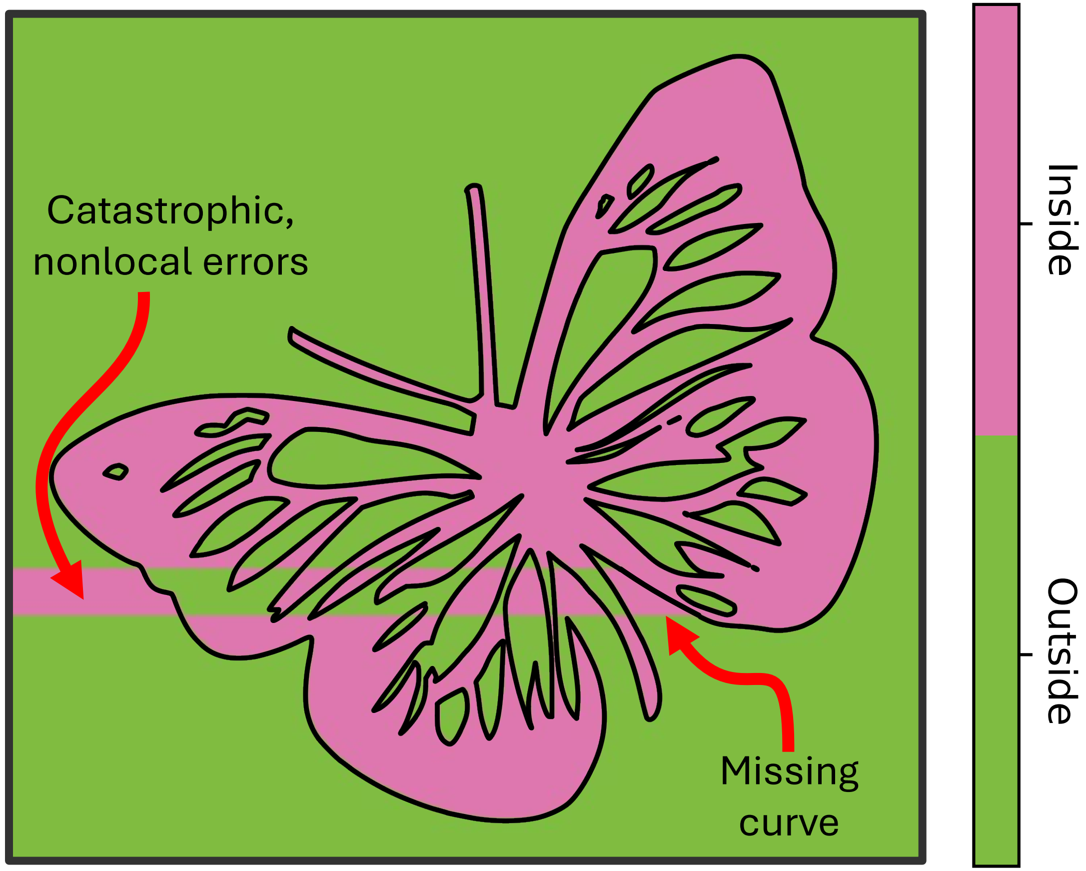
Outside Control Polygons

Additional Subdivisions for Nearer Points

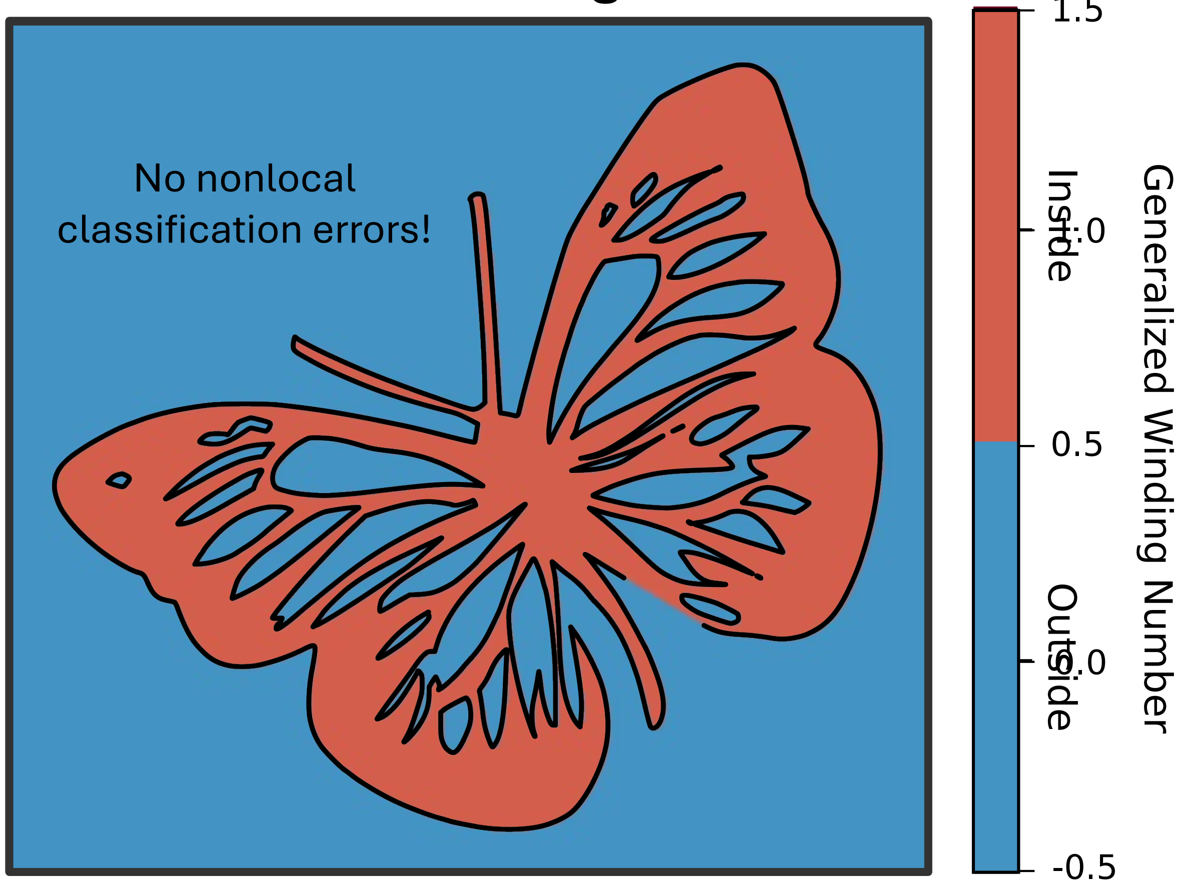
Intuitive Containment via Winding Numbers

- Unlike ray-casting, winding numbers are robust to geometric errors

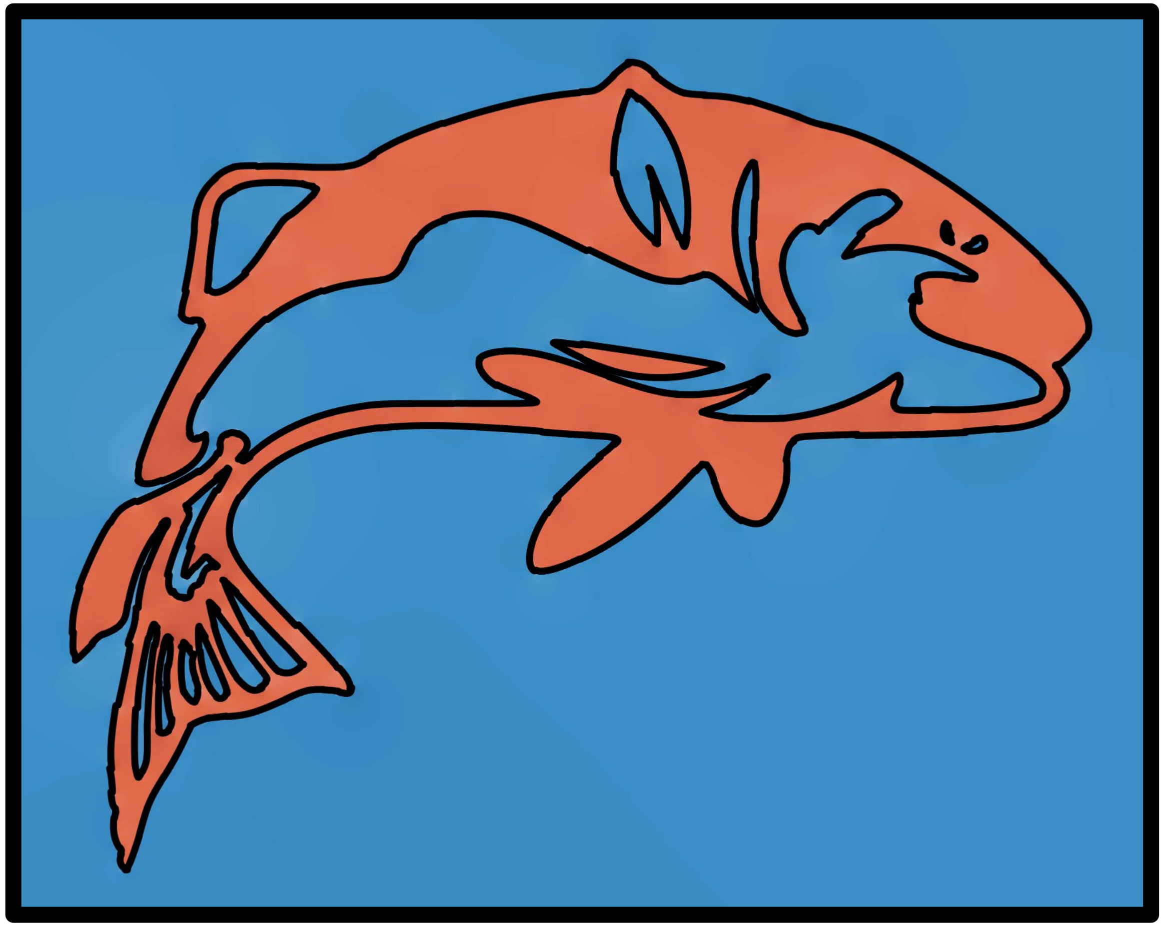
Simple Ray-Casting



Generalized Winding Numbers

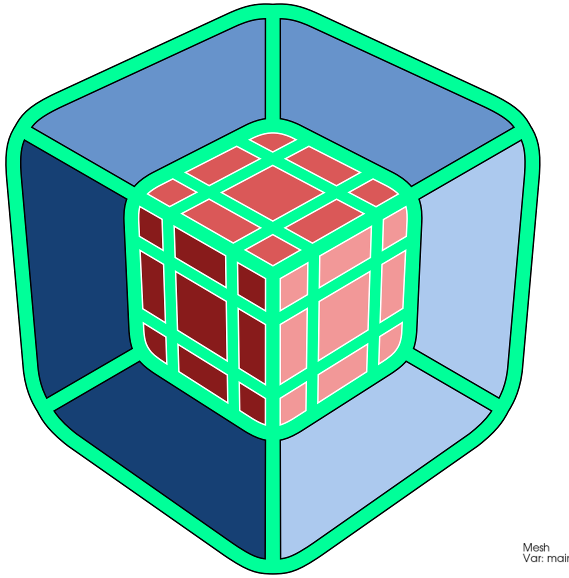


Intuitive Containment via Winding Numbers



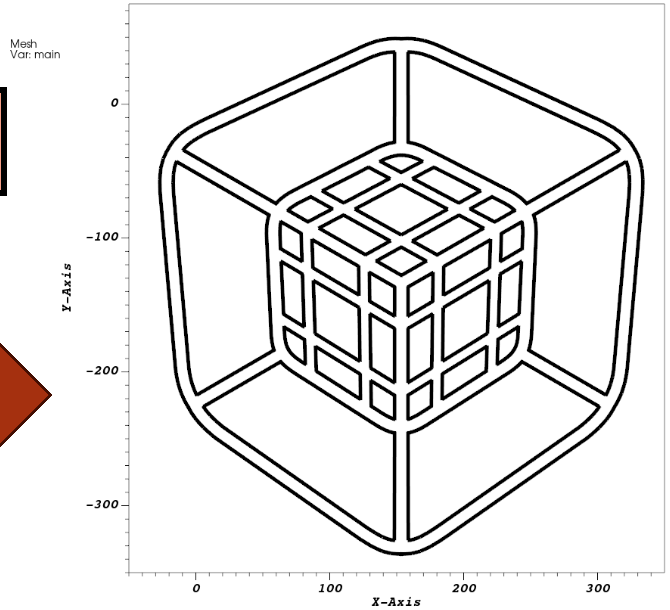
- Independent of the **frequency** of errors
- Does **not repair** the shape model itself

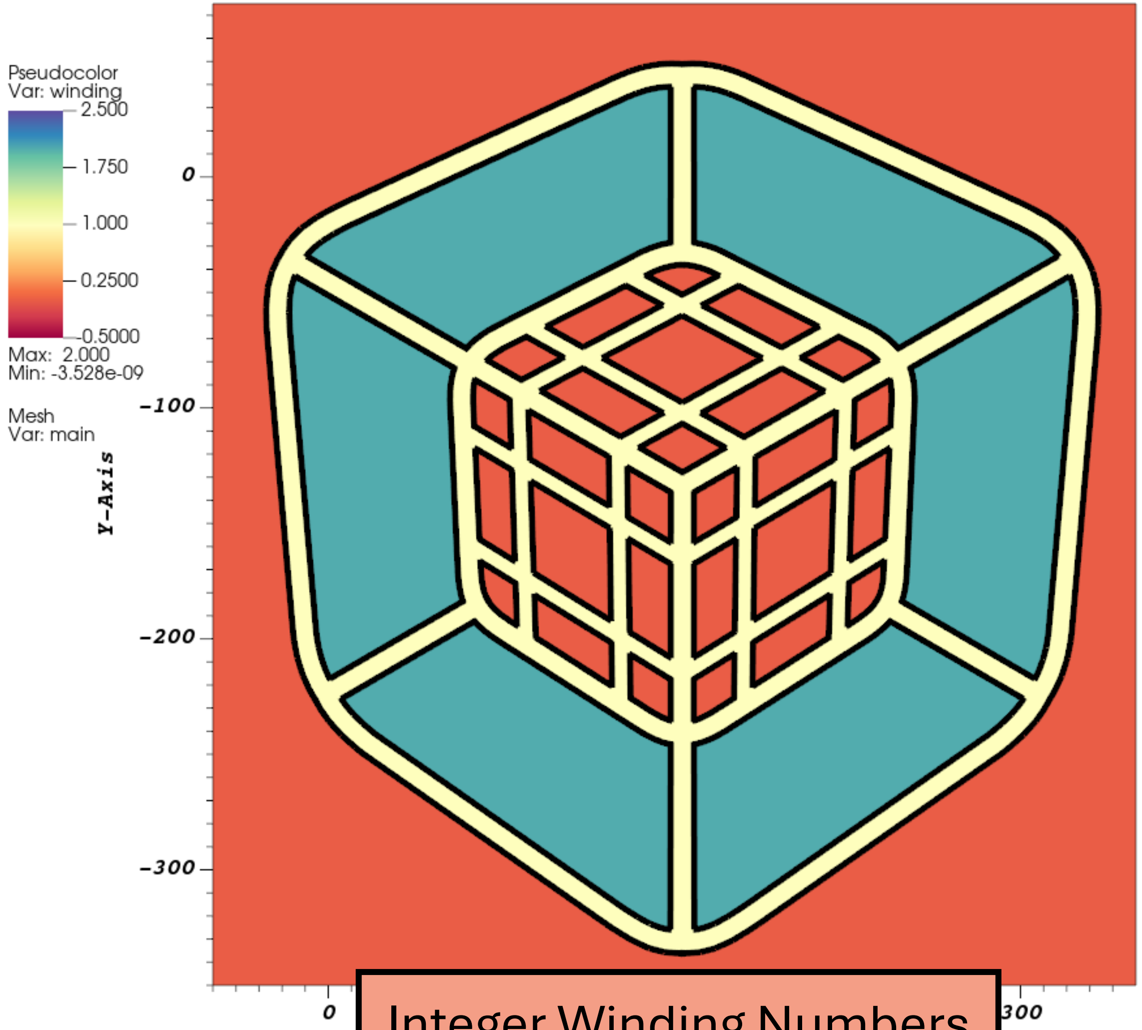
Example script in AXOM : Quest



SVG Curves

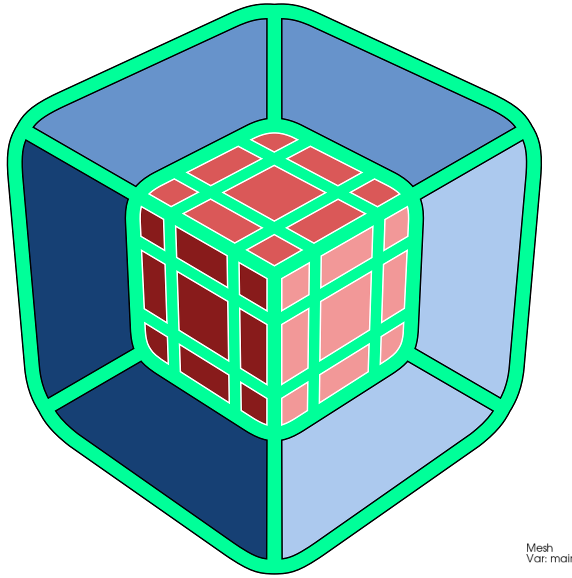
MFEM NURBS Mesh





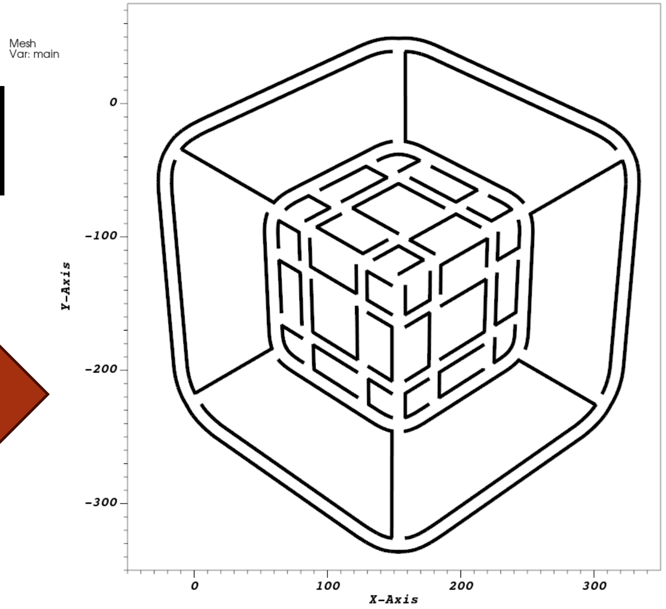
Integer Winding Numbers

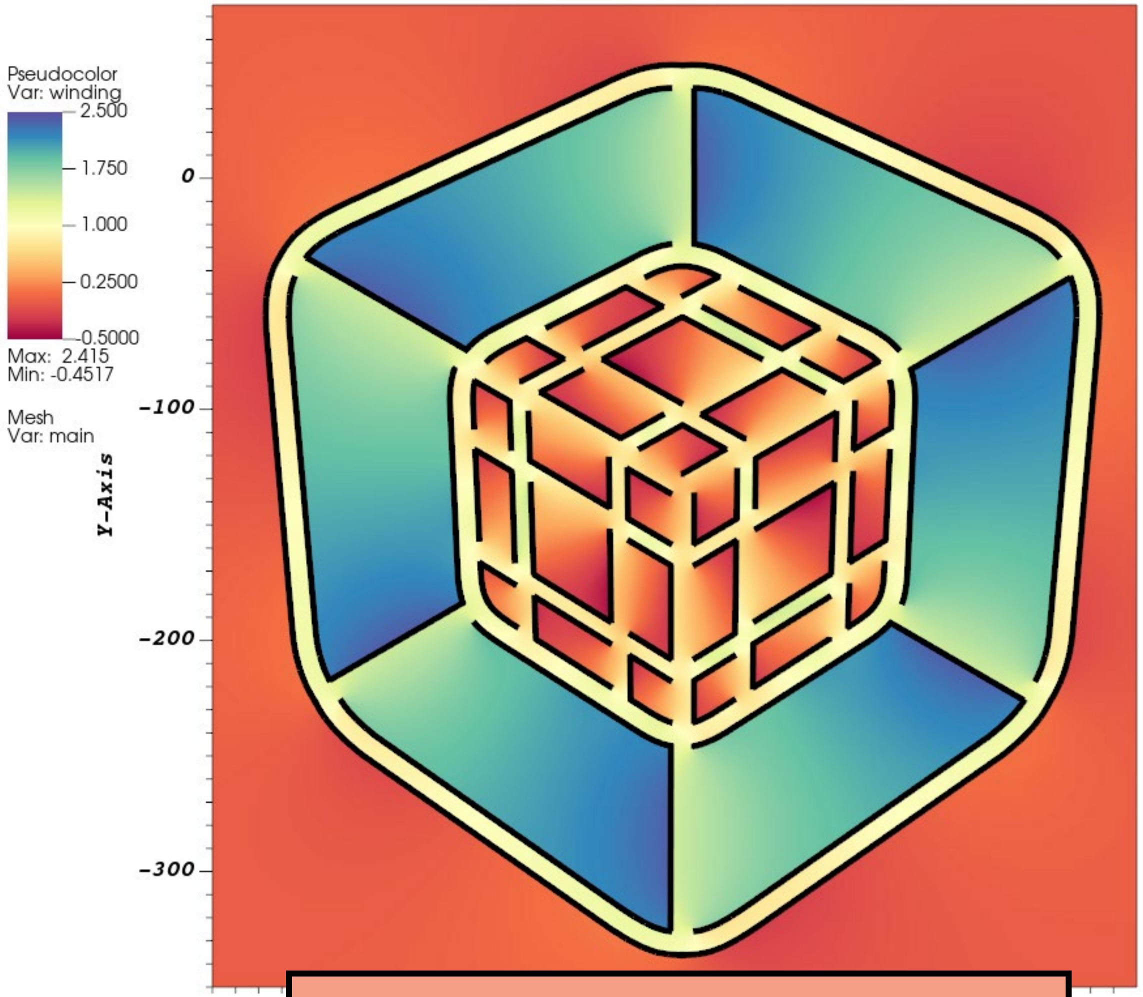
Example script in AXOM : Quest



SVG Curves

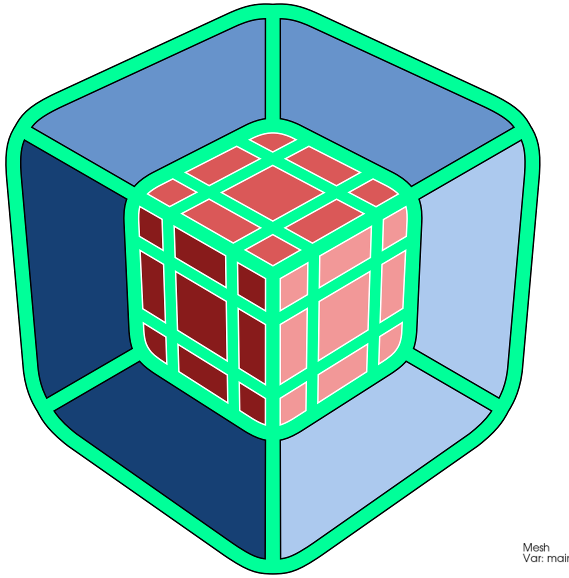
MFEM NURBS Mesh





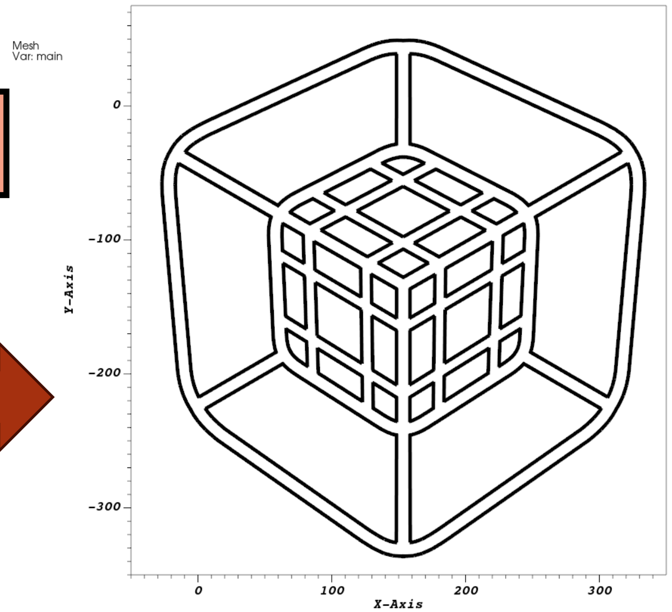
Generalized Winding Numbers

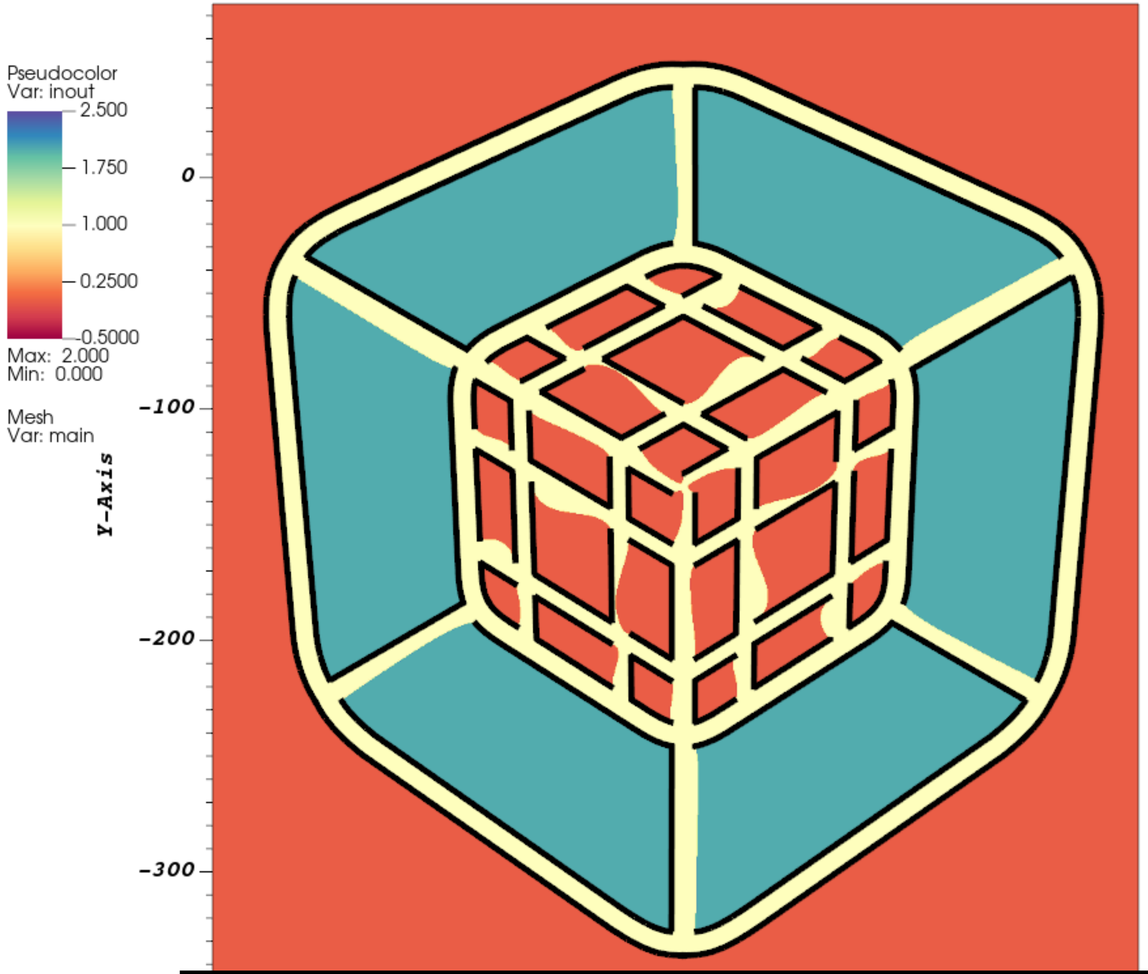
Example script in AXOM : Quest



SVG Curves

MFEM NURBS Mesh

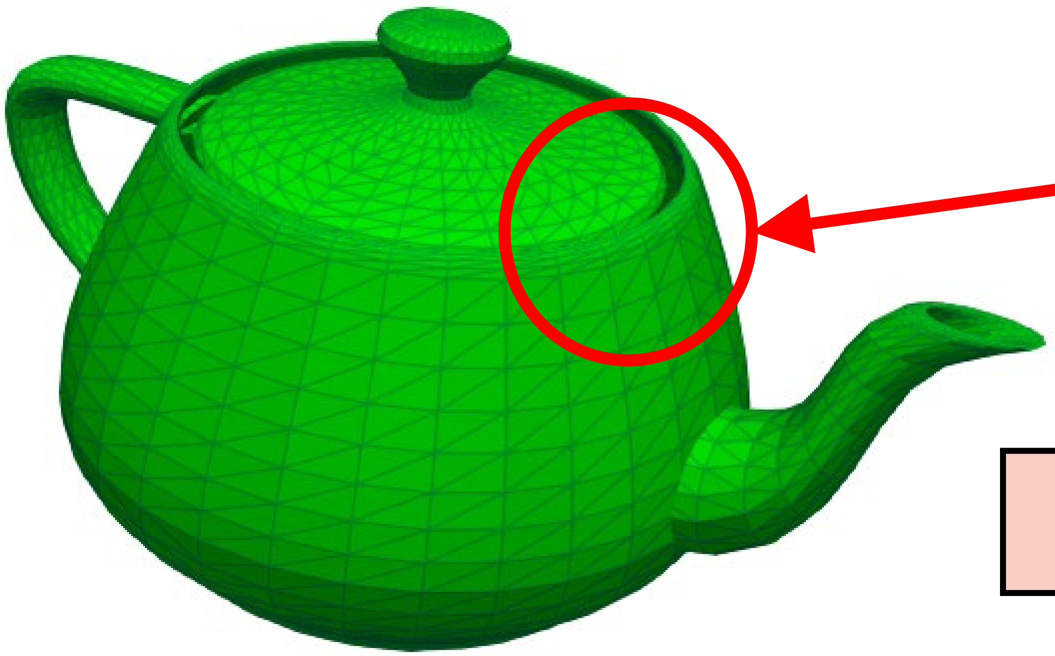




Rounded Generalized Winding Numbers

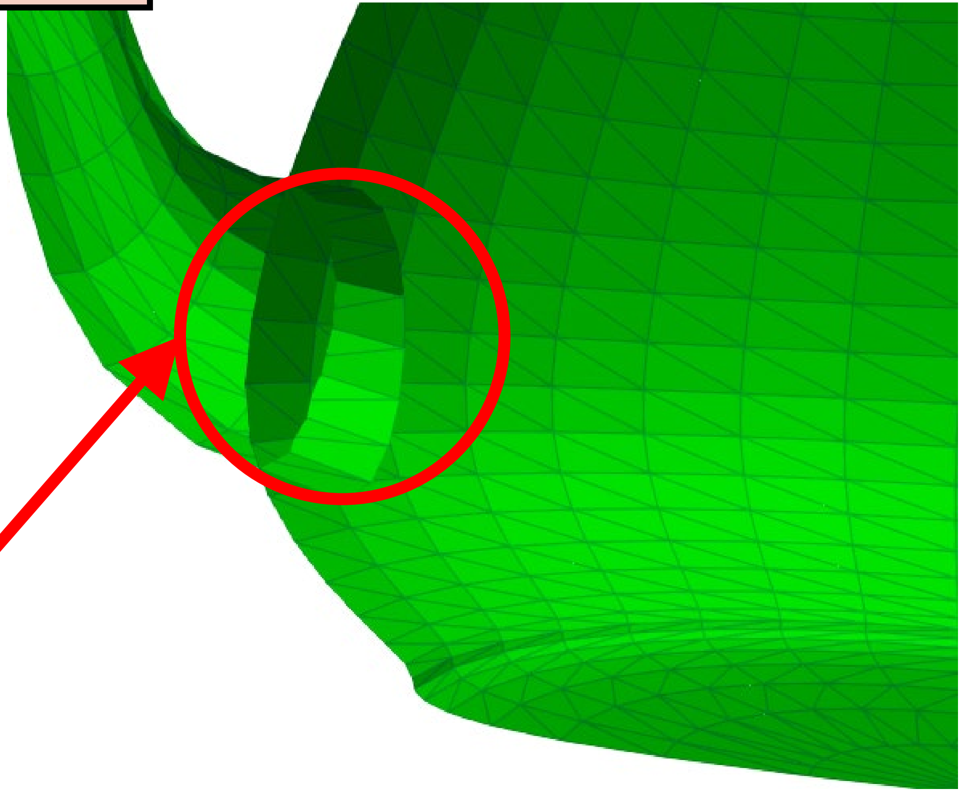
The Problem of "Messy" STL & CAD Geometry

In-the-wild 3D geometry is even worse!



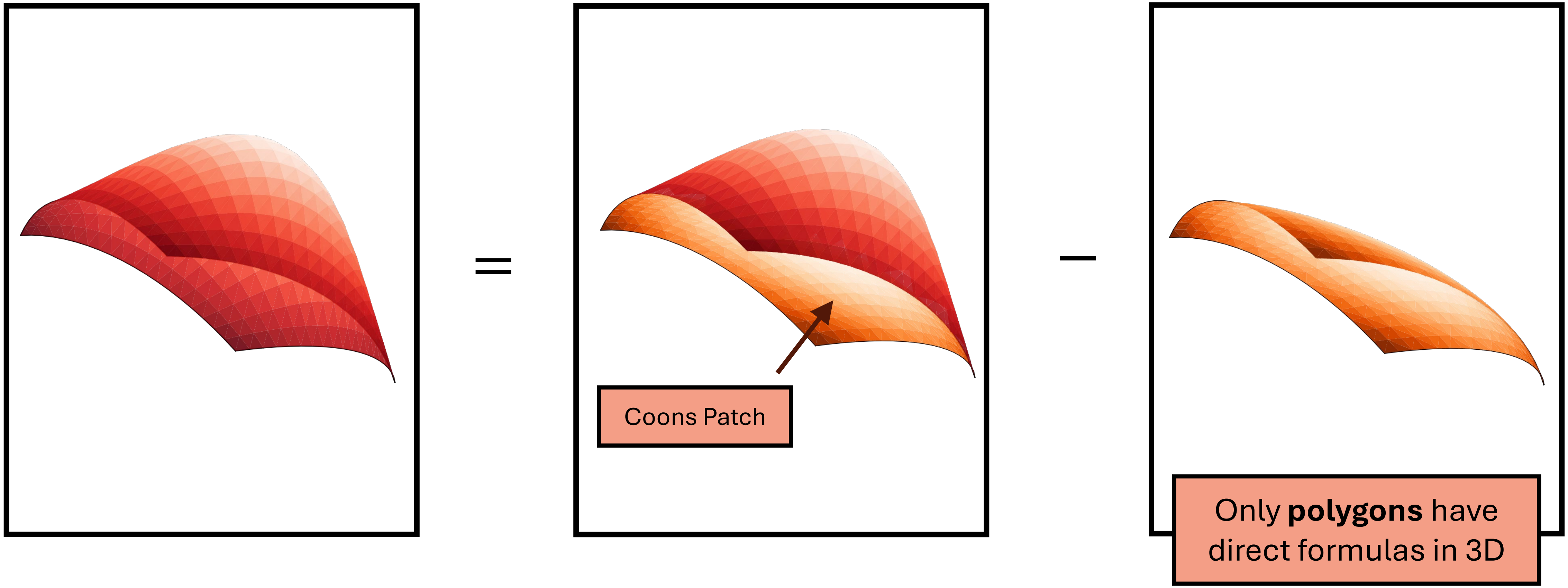
Gaps

Overlaps

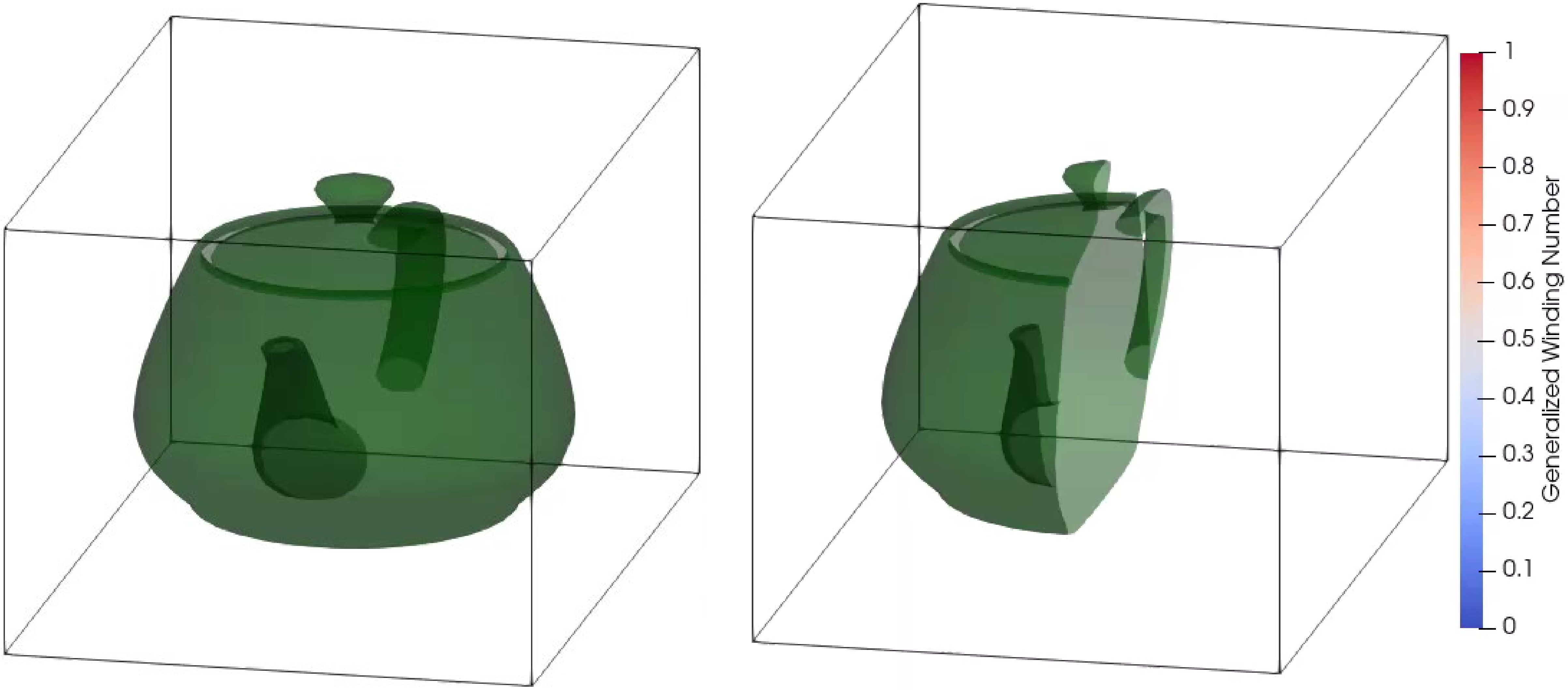


The analogous procedure doesn’t work in 3D!

Both are equally difficult!



Generalized Winding Numbers for "Messy" STL Data



Formulation with Stokes Theorem

Stokes theorem turns integrals on a surface into integrals on their boundary*

$$\iint_S (\nabla \times \mathbf{F}) \cdot d\boldsymbol{S} = \int_{\partial S} \boldsymbol{F} \cdot d\boldsymbol{r}$$

$$\nabla \times \mathbf{F} = \frac{\boldsymbol{x}}{\|\boldsymbol{x}\|^3}$$

$$\mathbf{F} = ???$$

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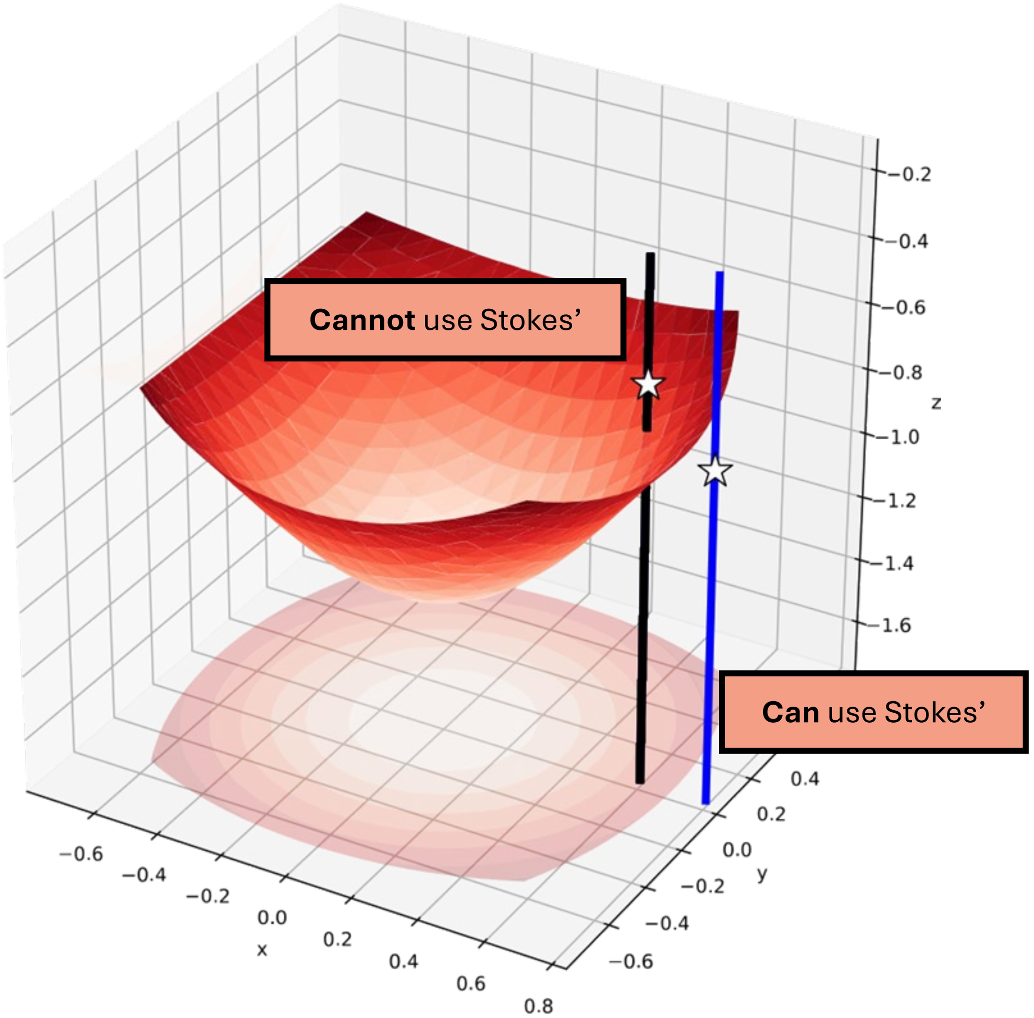
$$\nabla \times \mathbf{F} = \frac{\boldsymbol{x}}{\|\boldsymbol{x}\|^3}$$

$$\mathbf{F} = ???$$

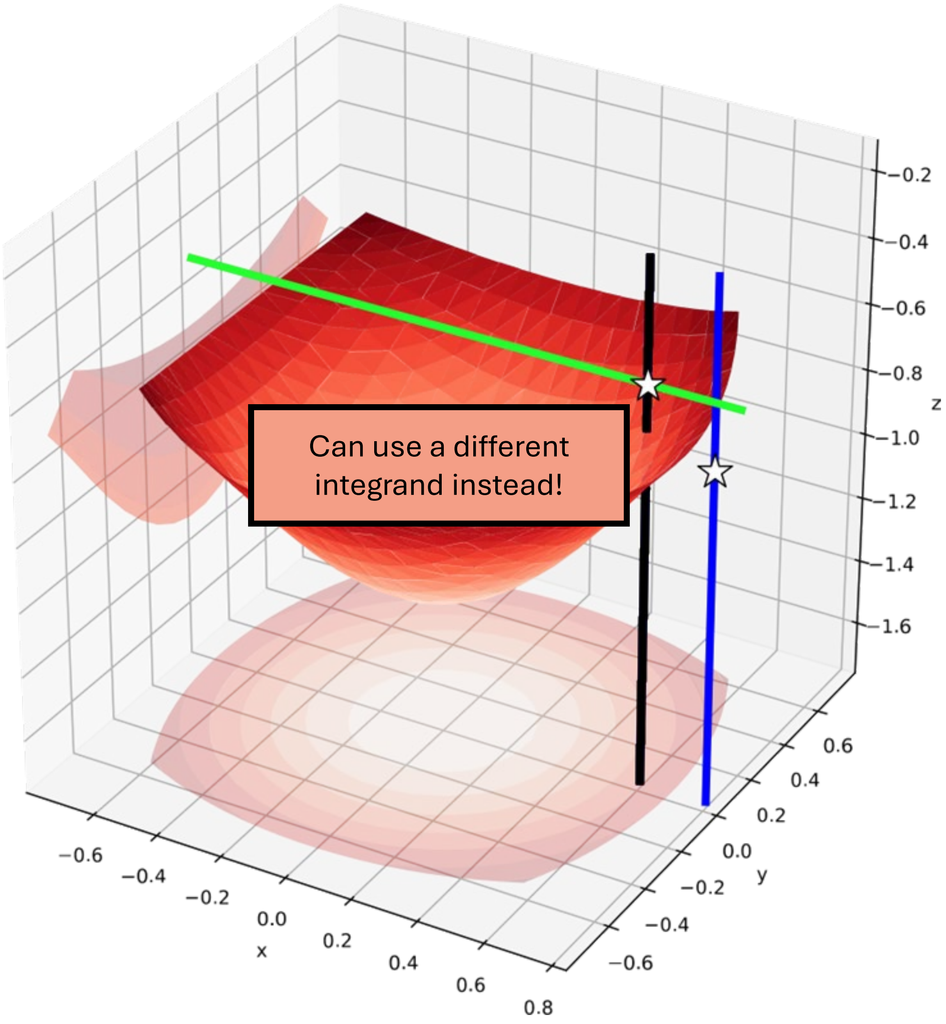
*Both $\nabla \times \mathbf{F}$ *and* $\mathbf{F}$ must be continuous on the surface!

Not every query can use the same integrand...



$$w = \frac{1}{4\pi}\int_{\partial S - \boldsymbol{q}} \left\langle \frac{yz}{(x^2+y^2)\|\boldsymbol{x}\|}, \frac{-xz}{(x^2+y^2)\|\boldsymbol{x}\|}, 0 \right\rangle \cdot \mathrm{d}\mathbf{r}$$

Not every query *has* to use the same integrand!



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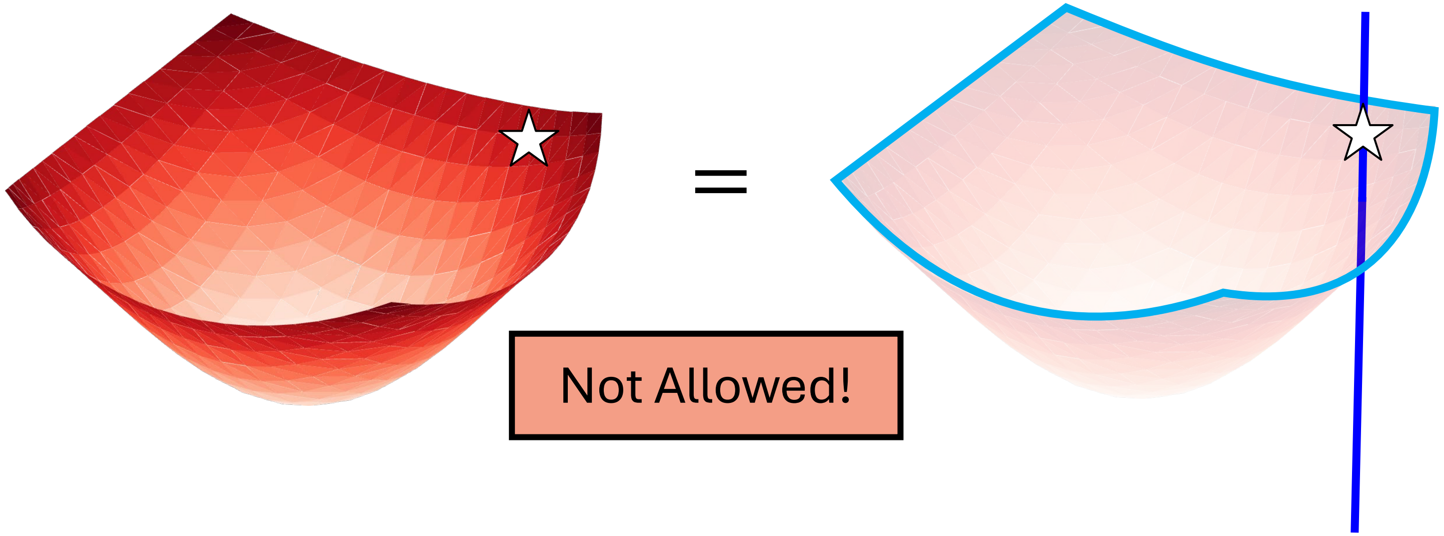
$$w = \frac{1}{4\pi}\int_{\partial S - \boldsymbol{q}} \left\langle \frac{-yz}{(x^2+z^2)\|\boldsymbol{x}\|}, 0, \frac{xy}{(x^2+z^2)\|\boldsymbol{x}\|} \right\rangle \cdot d\mathbf{r}$$

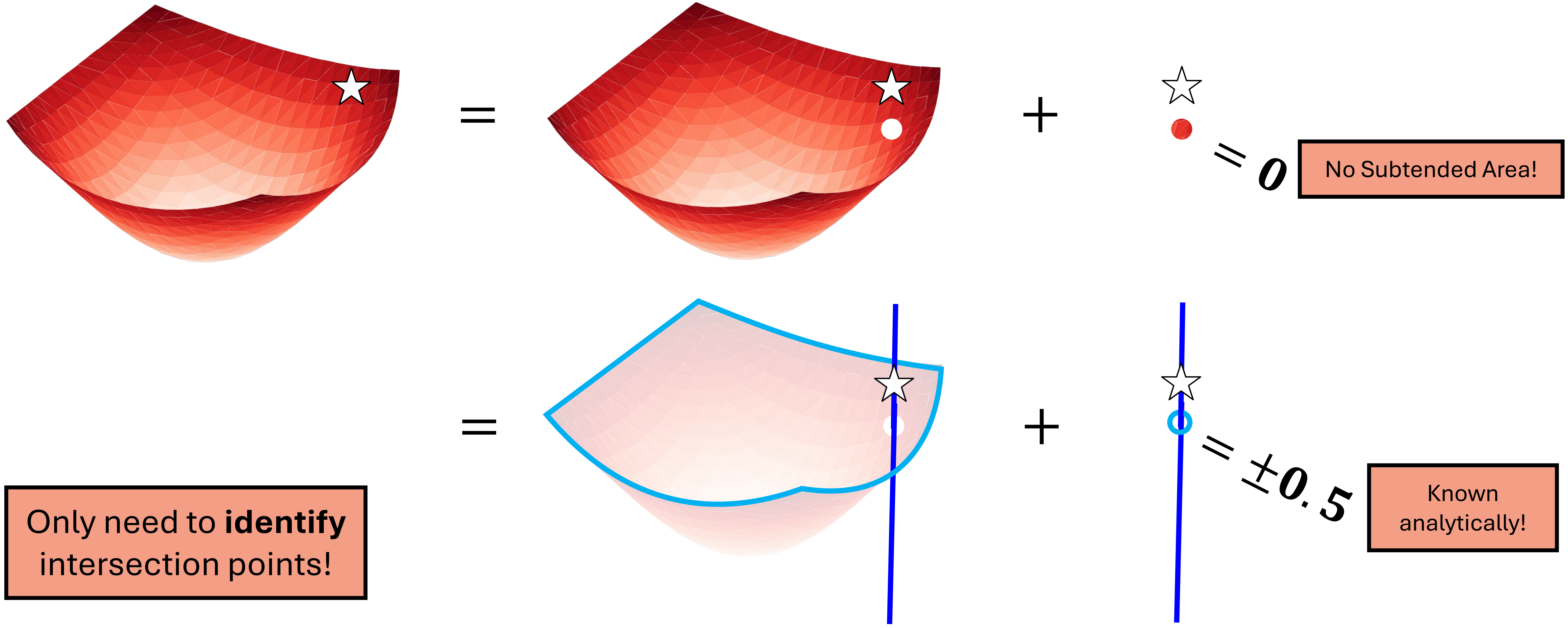
$$w = \frac{1}{4\pi}\int_{\partial S - \boldsymbol{q}} \left\langle 0, \frac{xz}{(y^2+z^2)\|\boldsymbol{x}\|}, \frac{-xy}{(y^2+z^2)\|\boldsymbol{x}\|} \right\rangle \cdot d\mathbf{r}$$

Handle near-surface points with analytic formula



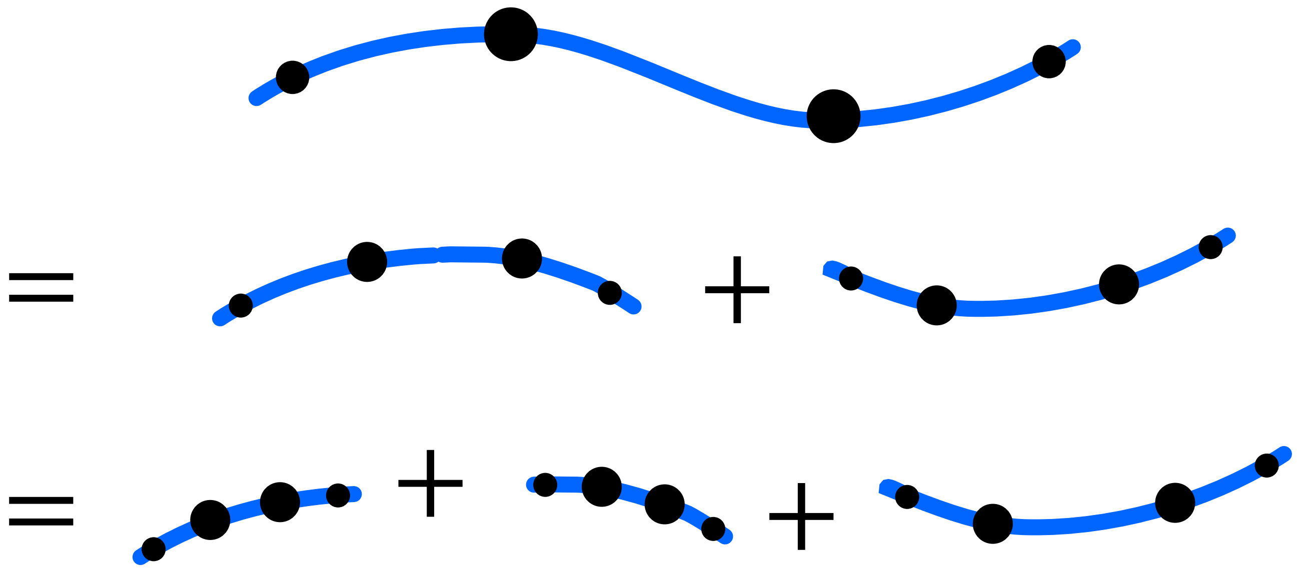
Handle near-surface points with analytic formula

- Can imagine (infinitesimal) disks cut out from the surface



Compute 1D line integrals with adaptive quadrature

- Using a geometrically adaptive quadrature provides verifiable results!

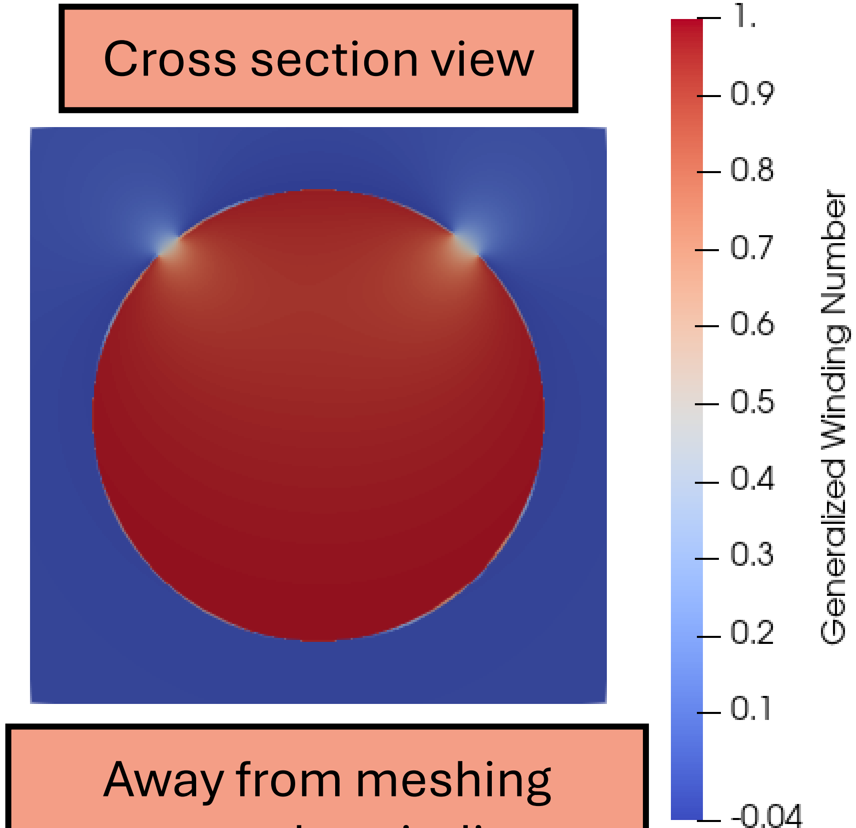


Crude, but effective!

Results: Exact Winding Numbers for Curved Surfaces

Slices of the GWN field

Cross section view

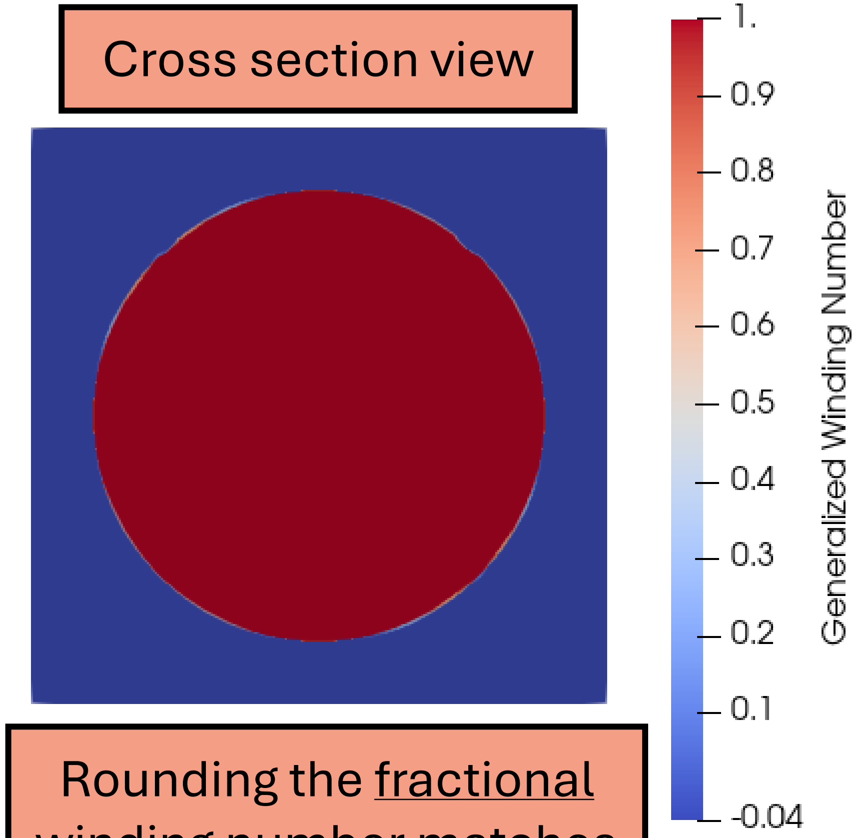


Away from meshing errors, the winding number is close to integer

Results: Containment for Messy Shapes

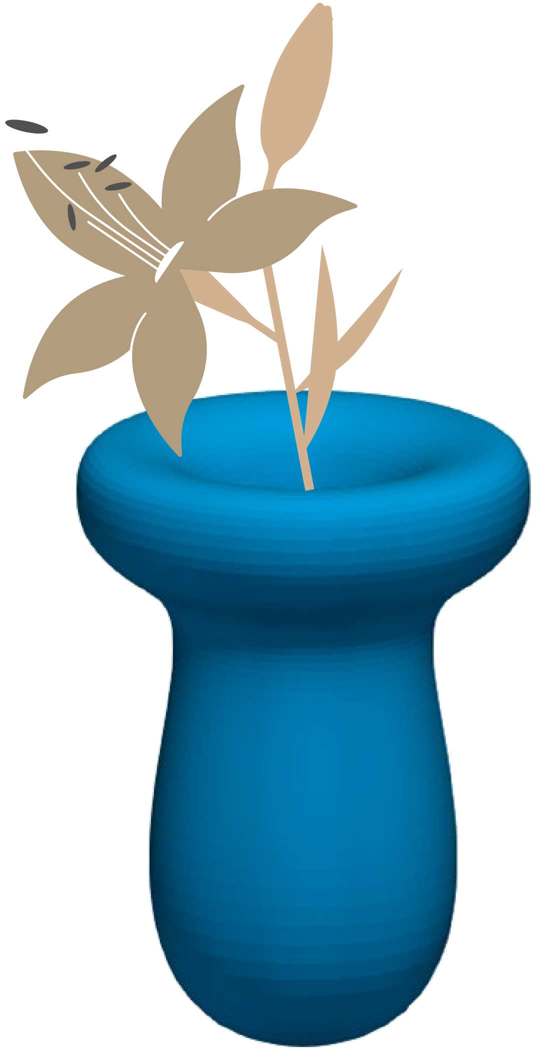
Slices of the GWN field

Cross section view

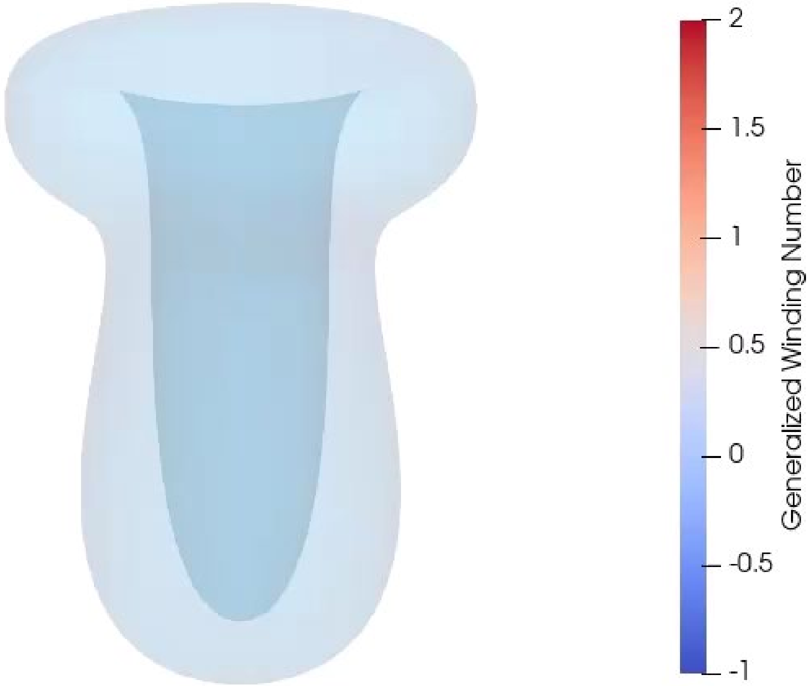


Rounding the fractional winding number matches user expectation

Results: Watertight Rational Patches

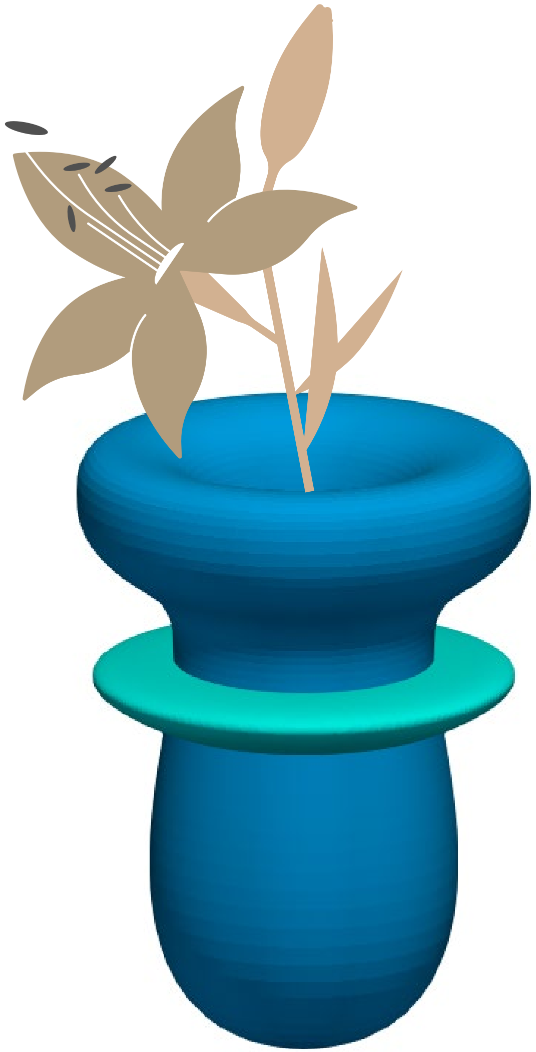


Surface composed of 4^{th} and 5^{th} order rational Bézier curves rotated around the central axis

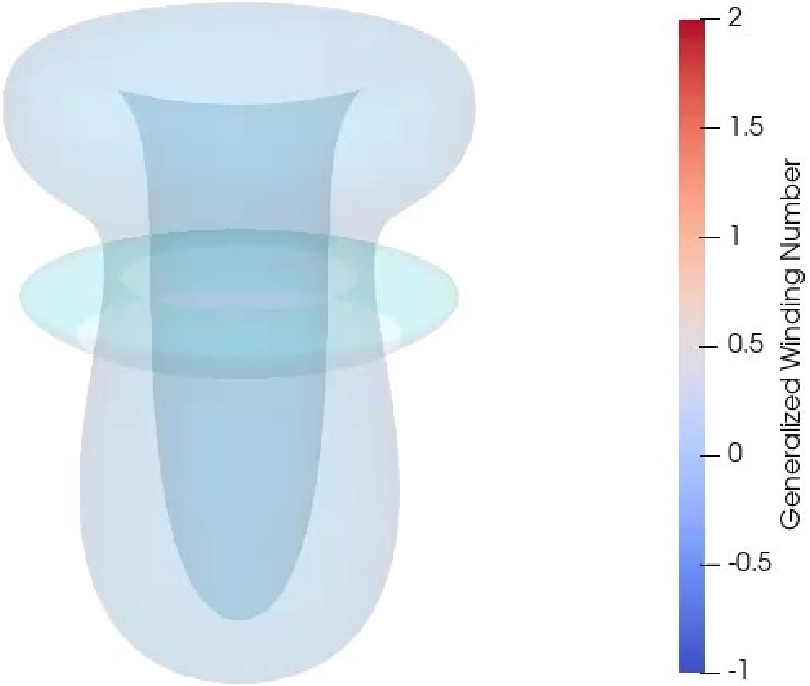


Aside: The 2D intersection curves of the slice are **really** difficult to compute analytically!

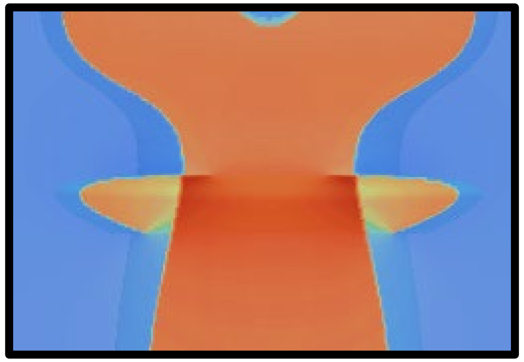
Results: Nonwatertight Rational Patches



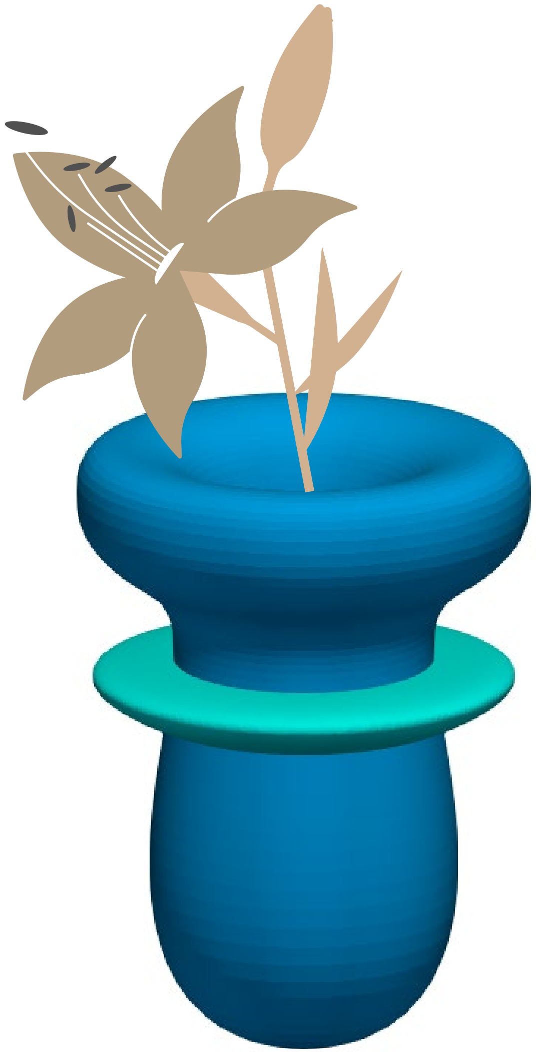
The interior of this ring *looks* closed, but it isn't!
This could cause **unexpected failure** during shaping.



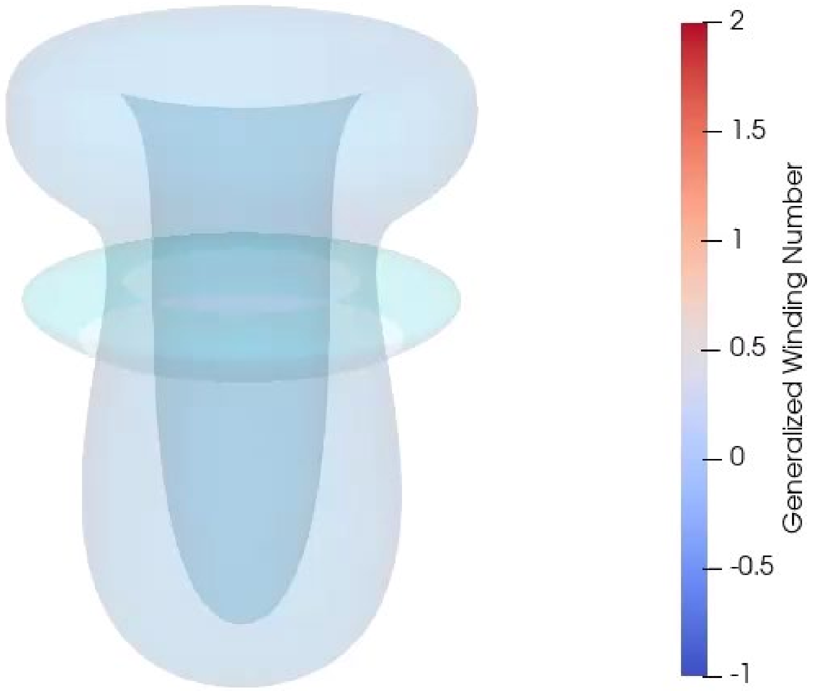
Cross Section



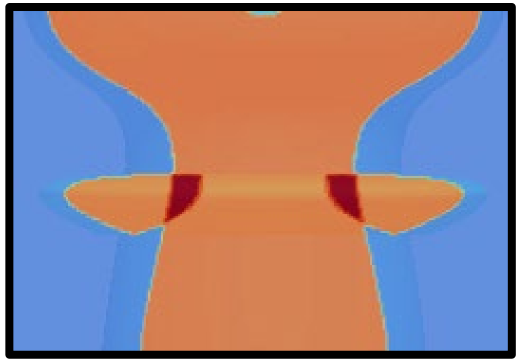
Results: Nonwatertight Rational Patches



Can still make in/out determination by rounding, and applying the Nonzero or EvenOdd rule



In either case, a winding number of zero means outside



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