SiMPL Method for Topology Optimization

Sigmoidal Mirror descent with Projected Latent variable

Dohyun Kim (Brown Univ., USA)

October 23, 2024

MFEM Workshop, LLNL

with Brendan Keith (Brown Univ., USA)
Thomas M. Surowiec (Simula, Norway)
Boyan S. Lazarov (LLNL, USA)

BROWN | Applied Mathematics

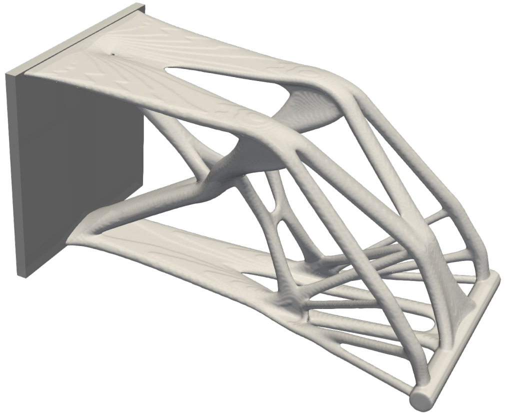
Outline

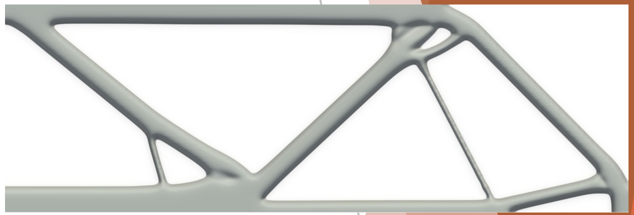
- Topology Optimization
- Gradient Descent and Mirror Descent
- Backtracking Line Search Algorithm
- Numerical Results

- Compliance Minimization
- Three Field Formulation
- Gradient descent method
- Mirror Descent

Topology Optimization

Topology Optimization





- The hello world problem: Compliance minimization
- We use Solid Isotropic Material Penalization (SIMP) and Helmholtz type filter

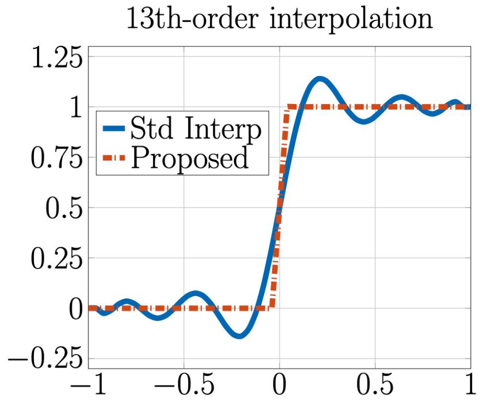
$$
\begin{aligned}
\underset{\rho}{\text{minimize}} \quad & \int_\Omega f \cdot u \, \mathrm{d}x \\
\text{subject to} \quad & -\operatorname{div}(r(\tilde{\rho})\mathrm{C} : \varepsilon(u)) = f \text{ in } \Omega, \\
& \text{B.C.s}, \\
& -\nabla(\epsilon^2 \nabla \tilde{\rho}) + \tilde{\rho} = \rho \text{ in } \Omega, \\
& \partial_n \tilde{\rho} = 0, \\
& 0 \le \rho \le 1, \\
& \int_\Omega \rho \, \mathrm{d}x \le \theta_M |\Omega|.
\end{aligned}
$$

- Here, $r(\tilde{\rho}) = \rho_0 + (1 - \rho_0)\tilde{\rho}^p, \ \epsilon = r_{\min}/2\sqrt{3}$

u	Displacement
$\tilde{\rho}$	Filtered Density
ρ	Density
f	External Force
C	Elasticity Tensor

Goal of this Talk

- Introduce a new method, SiMPL, for TO with
  - Point-wise feasible solution at each iteration (even with high-order)
  - Easy to implement
  - No additional memory (variable / large system solve)
  - Fast convergence



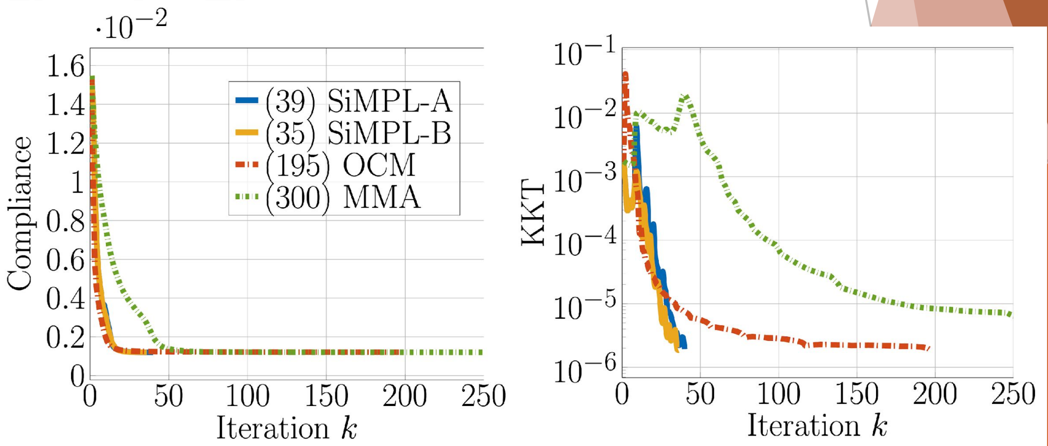
$$\psi_{k+1} = \psi_k - \alpha_k \nabla F(\rho_k) + \mu_{k+1}$$

Example 37: Topology Optimization

This example code solves a classical cantilever beam topology optimization problem. The aim is to find an optimal material density field ϱ in $L^1(\Omega)$ to minimize the elastic compliance; i.e.,

$$\text{minimize} \int_\Omega \mathbf{f} \cdot \mathbf{u}(\varrho)\, dx \;\text{ over }\; \varrho \in L^1(\Omega)$$

$$\text{subject to } 0 \le \varrho \le 1 \;\text{ and }\; \int_\Omega \varrho\, dx = \theta \,\mathrm{vol}(\Omega)\,.$$



Projected Gradient Descent Method

- The unconstrained gradient descent method

$$
\begin{aligned}
\rho_{k+1} &= \rho_k - \alpha_k \nabla F(\rho_k) \\
&= \operatorname*{argmin}_{\rho} F(\rho_k) + \int_\Omega \nabla F(\rho_k) \cdot (\rho - \rho_k) dx + \frac{1}{2\alpha_k} \|\rho - \rho_k\|^2_{L^2(\Omega)} \\
&= \operatorname*{argmin}_{\rho} \int_\Omega \nabla F(\rho_k) \cdot \rho dx + \frac{1}{2\alpha_k} \|\rho - \rho_k\|^2_{L^2(\Omega)}
\end{aligned}
$$

- For constrained minimization problem,
  we obtain a "Projected" gradient method

$$
\begin{aligned}
\rho_{k+1} &= \operatorname*{argmin}_{\rho \in \mathcal{C}} \int_\Omega \nabla F(\rho_k) \cdot \rho dx + \frac{1}{2\alpha_k} \|\rho - \rho_k\|^2_{L^2(\Omega)} \\
&= \mathcal{P}_{\mathcal{C}}(\rho_k - \alpha_k \nabla F(\rho_k))
\end{aligned}
$$

Mirror Descent Method

- Mirror descent is a generalized gradient descent method

$$\rho_{k+1,GD} = \operatorname*{argmin}_{\rho\in\mathcal{C}} \int_\Omega \nabla F(\rho_k)\cdot\rho dx + \frac{1}{\alpha_k}\frac{1}{2}\|\rho-\rho_k\|^2_{L^2(\Omega)}$$

$$\rho_{k+1,MD} = \operatorname*{argmin}_{\rho\in\mathcal{C}} \int_\Omega \nabla F(\rho_k)\cdot\rho dx + \frac{1}{\alpha_k}D_\varphi(\rho,\rho_k)$$

$$= \mathcal{P}_\mathcal{C}((\nabla\varphi)^{-1}(\nabla\varphi(\rho_k)-\alpha_k\nabla F(\rho_k)))$$

$$\mathcal{P}_\mathcal{C}(\rho) = \operatorname*{argmin}_{q\in\mathcal{C}} D_\varphi(q,\rho)$$

- The Bregman divergence is a generalized squared distance
  - $D_\varphi(\rho,q) = \varphi(\rho)-\varphi(q)-\int_\Omega \nabla\varphi(q)\cdot(\rho-q)dx$
  - If φ is strictly (or strongly) convex, then so is $D_\varphi(\cdot,q)$
  - Not symmetric, No triangle inequality

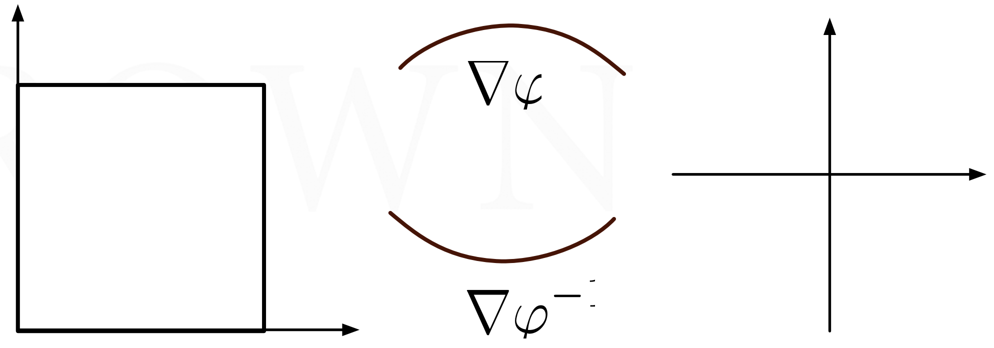
Bregman Divergence and Mapping

- Fermi-Dirac entropy $\varphi(\rho) = \int_\Omega \rho \log(\rho) + (1-\rho)\log(1-\rho)dx$

$$\nabla\varphi(\rho) = \log\left(\frac{\rho}{1-\rho}\right)$$

$$(\nabla\varphi)^{-1}(\psi) = \frac{1}{1+\exp(-\psi)}$$

$$= \sigma(\psi$$



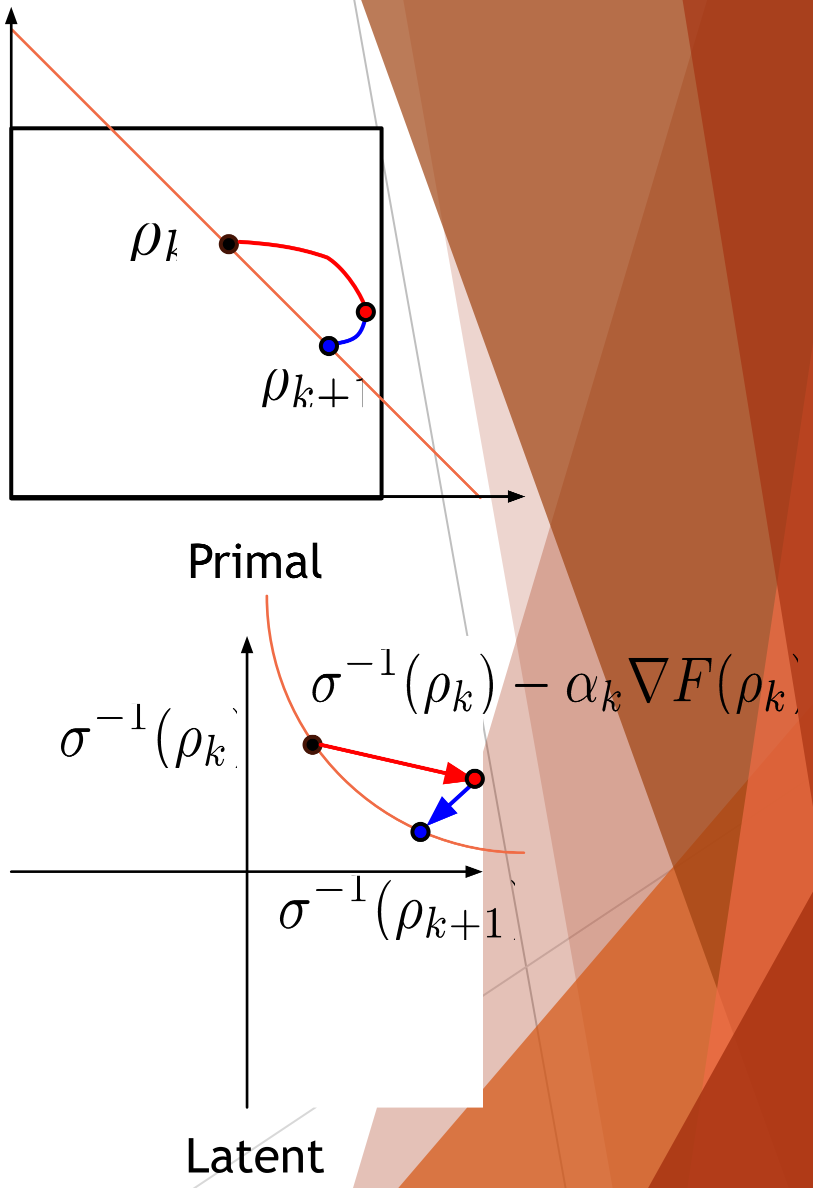
Projected Mirror Descent

- The projected mirror descent

$$\begin{aligned}\rho_{k+1} &= \mathcal{P}_{\mathcal{C}}((\nabla\varphi)^{-1}(\nabla\varphi(\rho_k) - \alpha_k \nabla F(\rho_k))) \\ &= \mathcal{P}_{\mathcal{C}}(\sigma(\sigma^{-1}(\rho_k) - \alpha_k \nabla F(\rho_k))) \\ &= \sigma(\sigma^{-1}(\rho_k) - \alpha_k \nabla F(\rho_k) + \mu_{k+1})\end{aligned}$$

Here, $\mu_{k+1} \in \mathbb{R}$ solves the volume correction equation

$$\int_\Omega \sigma(\sigma^{-1}(\rho_k) - \alpha_k \nabla F(\rho_k) + \mu_{k+1})dx = \theta|\Omega|$$



SiMPL – Latent Variable

- Introduce the latent variable $\psi = \sigma^{-1}(\rho)$

$$\rho_{k+1} = \sigma(\sigma^{-1}(\rho_k) - \alpha_k \nabla F(\rho_k) + \mu_{k+1})$$

$$\sigma^{-1}(\rho) = \log\left(\frac{\rho}{1-\rho}\right)$$

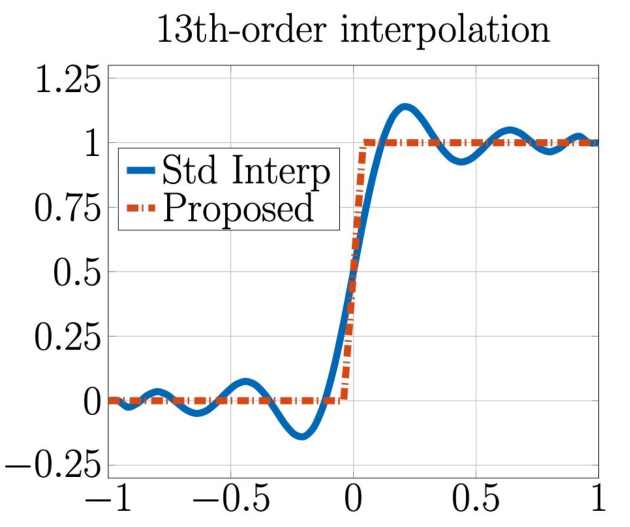
$$\sigma^{-1}(\rho_{k+1}) = \sigma^{-1}(\rho_k) - \alpha_k \nabla F(\rho_k) + \mu_{k+1}$$

$$\psi_{k+1} = \psi_k - \alpha_k \nabla F(\sigma(\psi_k)) + \mu_{k+1}$$

- We find a discrete approximation $\psi_{k,h}$

$$\rho_{k,h} = \sigma(\psi_{k,h})$$

$$\sigma(40) - \sigma(20) \approx 10^{-9}$$

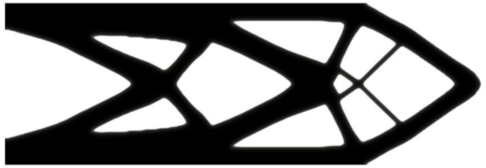


- Major benefits
  - Update step is linear in ψ with a scalar nonlinear equation
  - No logarithmic transform $\sigma^{-1}(x) = \log(x/(1-x))$
  - Bound constraint is satisfied point-wise

Backtracking Line Search

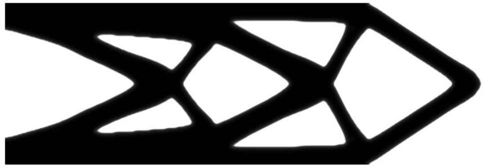
- Step size plays a key role to obtain an efficient and stable algorithm

$\alpha_k = 10k$



$F(\bar{\rho}_h) = 3.75 \cdot 10^{-3}$

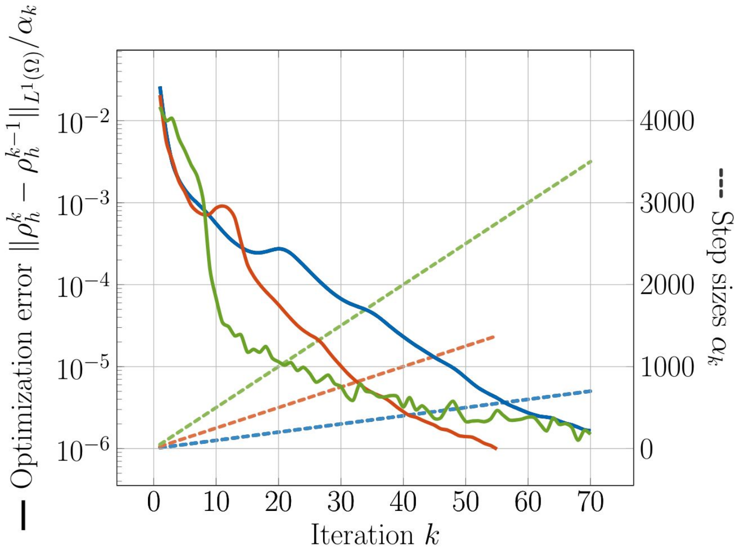
$\alpha_k = 25k$



$F(\bar{\rho}_h) = 3.72 \cdot 10^{-3}$

$\alpha_k = 50k$

$F(\bar{\rho}_h) \approx 7 \cdot 10^{-3}$



B. Keith and T. M. Surowiec, Proximal Galerkin: A structure preserving finite element method for pointwise bound constraints, arXiv:2307.12444v5, 2024

Backtracking Line Search

- Step size plays a key role to obtain an efficient and stable algorithm
- Convergence analysis may be carried out with "relative smoothness"

$$\left|\int_\Omega (\nabla F(x) - \nabla F(y))(x-y)dx\right| < L \int_\Omega (\nabla\varphi(x) - \nabla\varphi(y))(x-y)dx \quad \forall x, y \in C$$

$$L_k := \frac{\left|\int_\Omega (\nabla F(\rho_k) - \nabla F(\rho_{k-1}))(\rho_k - \rho_{k-1})dx\right|}{\int_\Omega (\psi_k - \psi_{k-1})(\rho_k - \rho_{k-1})dx}$$

$$\alpha_k := L_k^{-1}.$$

- We use a generalized Armijo rule (SiMPL-A) or Bregman rule (SiMPL-B) with the above initial step size

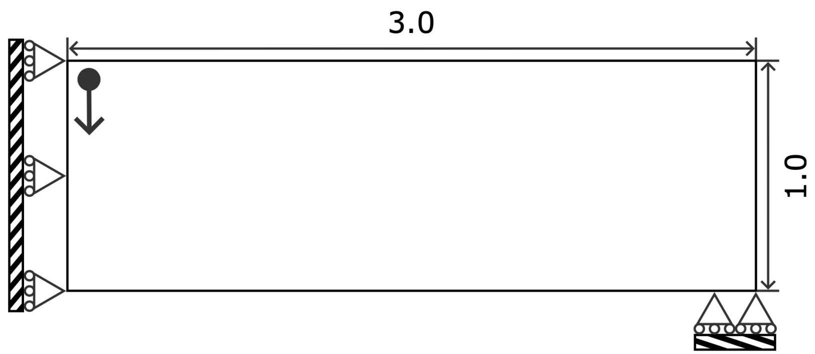
$$\begin{aligned} F(\boldsymbol{\rho}_{k+1,q}) \leq & F(\boldsymbol{\rho}_{k,q}) \\ & + c_1 \nabla F(\boldsymbol{\rho}_{k,q})^T \boldsymbol{M}_q (\boldsymbol{\rho}_{k+1,q} - \boldsymbol{\rho}_{k,q}) \end{aligned}$$

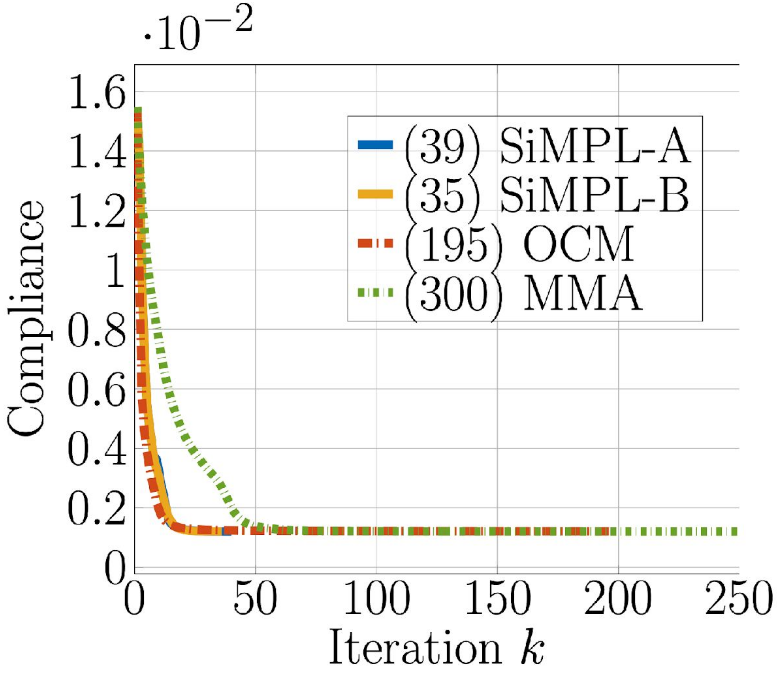
$$\begin{aligned} F(\boldsymbol{\rho}_{k+1,q}) \leq & F(\boldsymbol{\rho}_{k,q}) \\ & + \nabla F(\boldsymbol{\rho}_{k,q})^T \boldsymbol{M}_q (\boldsymbol{\rho}_{k+1,q} - \boldsymbol{\rho}_{k,q}) \\ & + \frac{1}{\alpha_k} D_\varphi(\boldsymbol{\rho}_{k+1,q}, \boldsymbol{\rho}_{k,q}) \end{aligned}$$

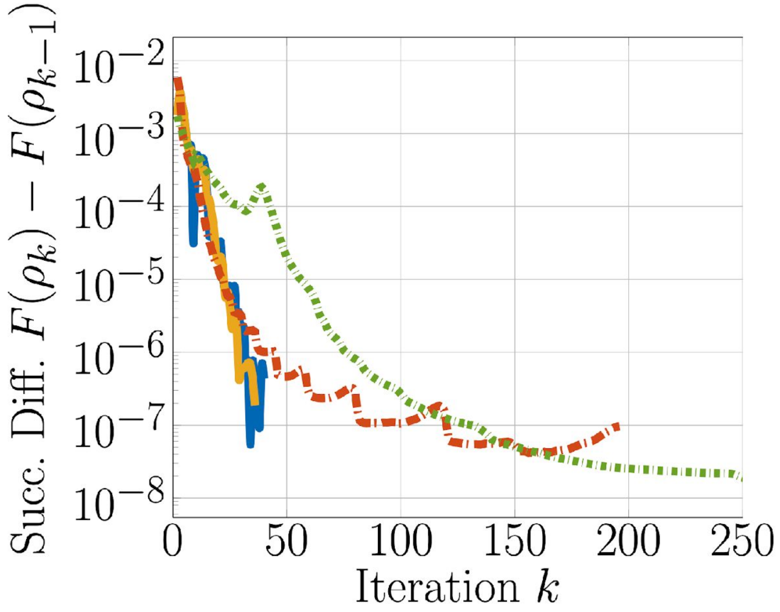
- Implemented in MFEM C++ Library
- Comparison with MMA and OCM
- Mesh independent behavior
- Beyond compliance minimization
  - Bridge Design
  - Compliant mechanism

Numerical Results

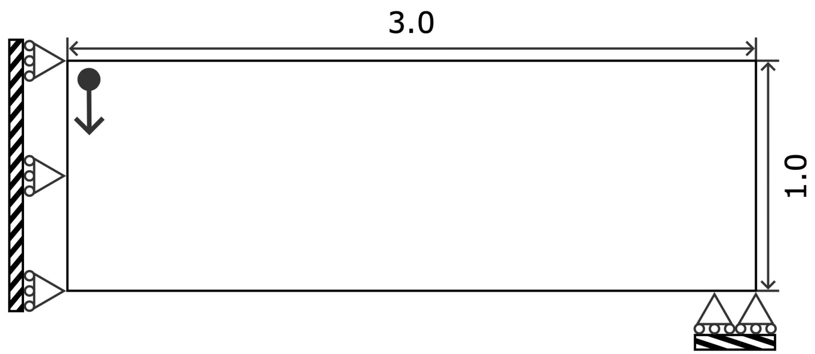
MBB Beam Problem with h=1/256



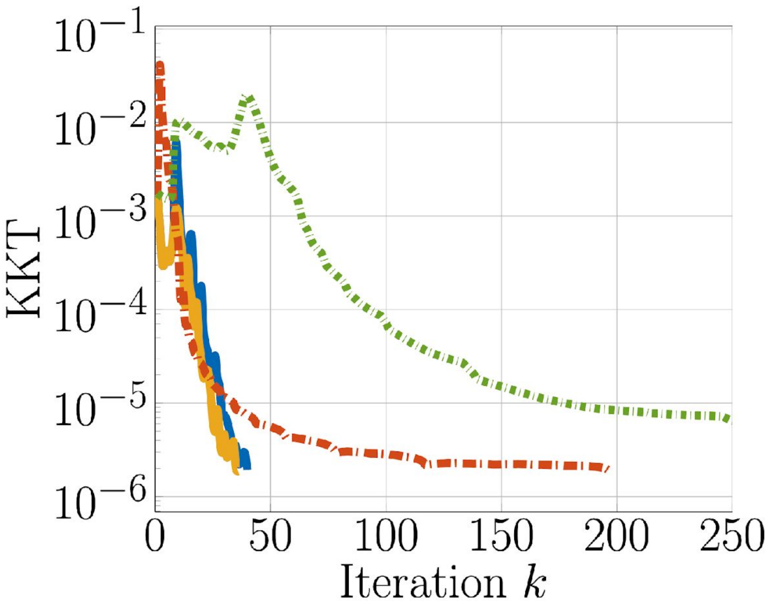


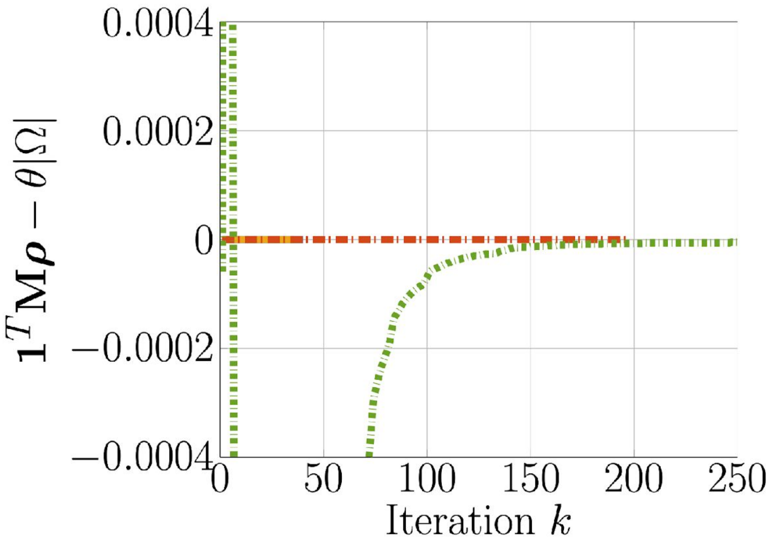


MBB Beam Problem with h=1/256

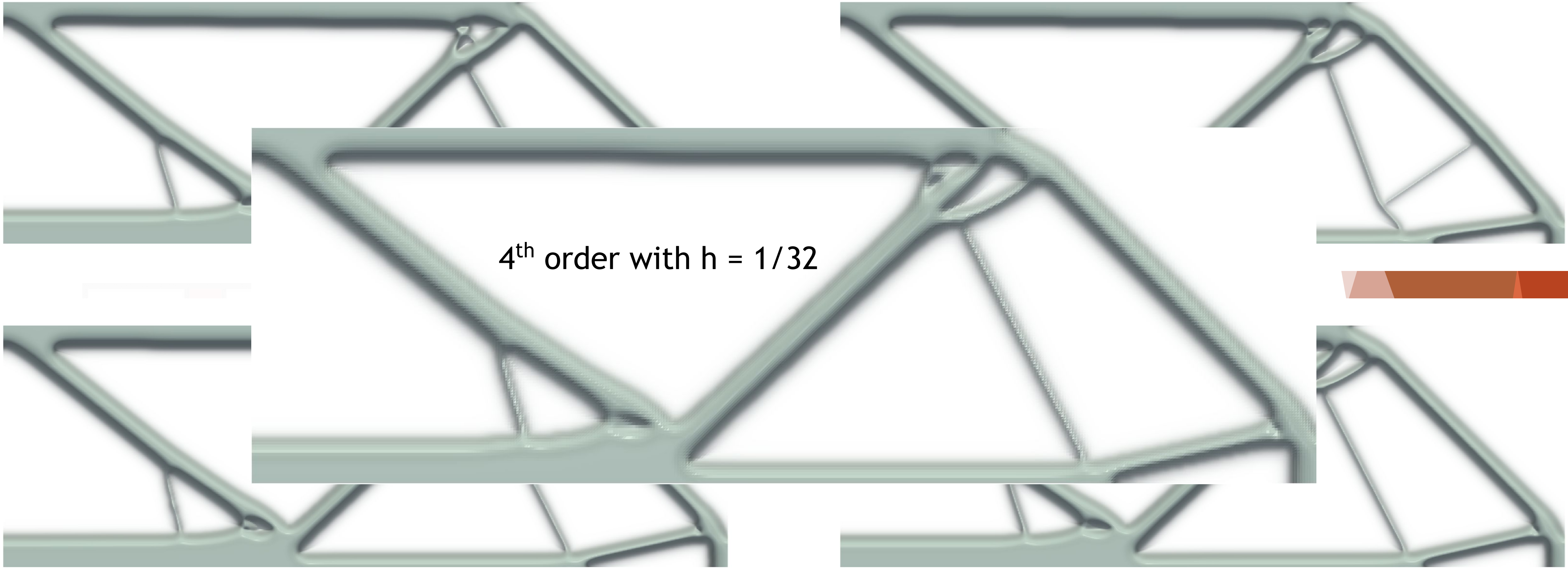


	$F(\rho^\star)$	KKT error	Iter.	Eval
SiMPL-A	1.2055×10^{-3}	3.63×10^{-6}	36	39
SiMPL-B	1.2083×10^{-3}	8.74×10^{-7}	36	50
OC	1.2129×10^{-3}	1.93×10^{-6}	195	195
MMA	1.2054×10^{-3}	4.16×10^{-3}	300	300





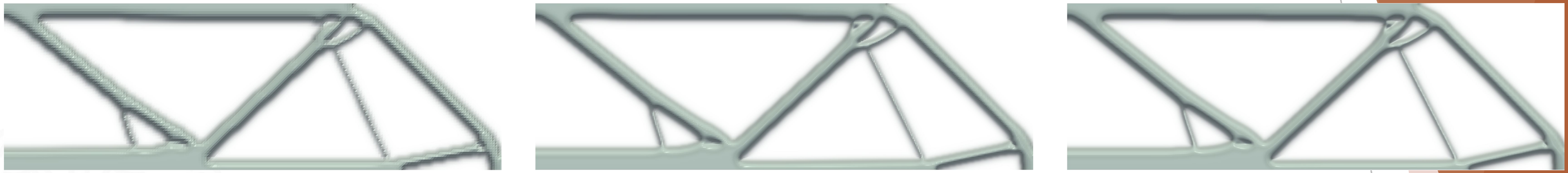
MBB Beam Problem with h=1/256

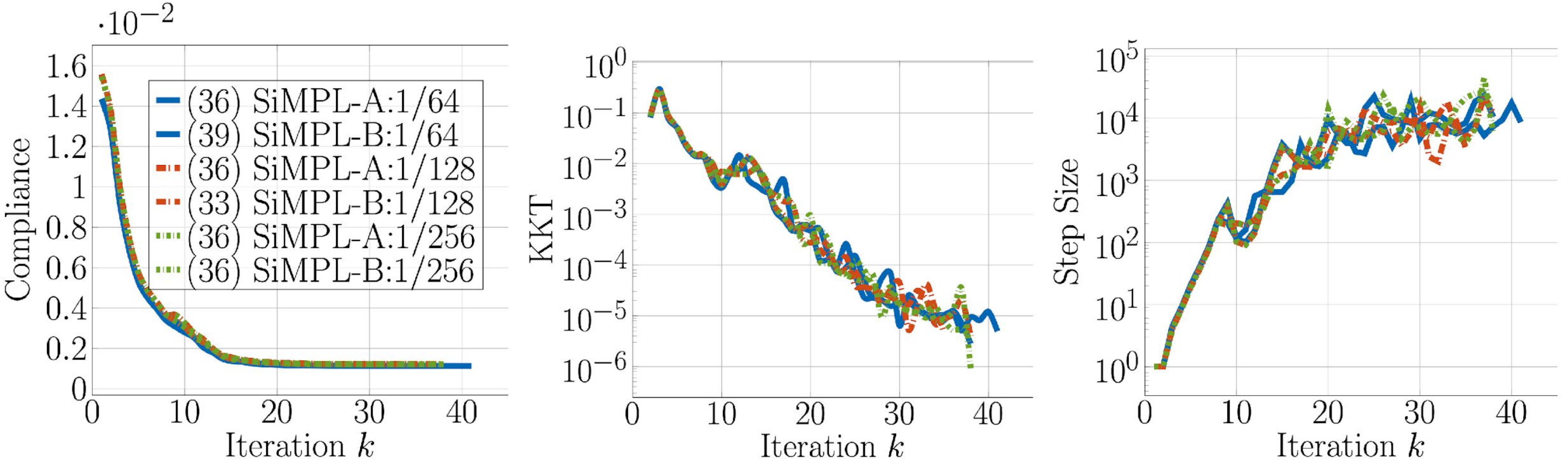


SiMPL-A

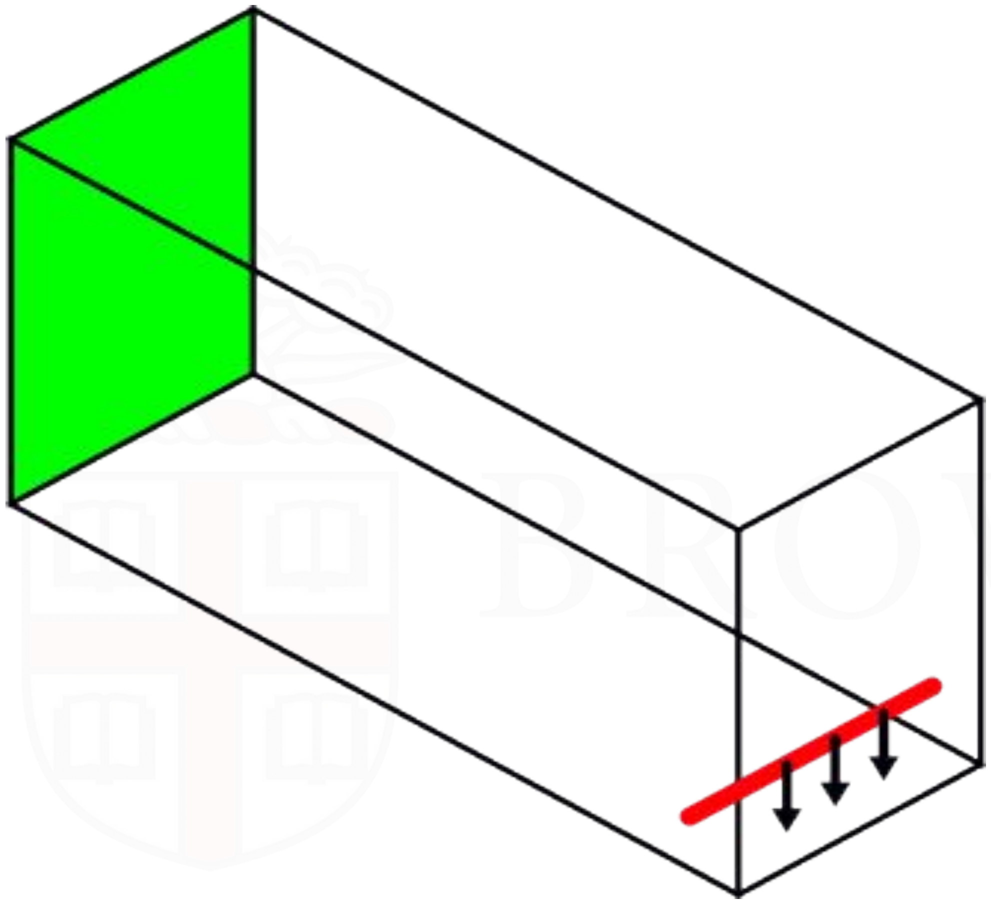
SiMPL-B

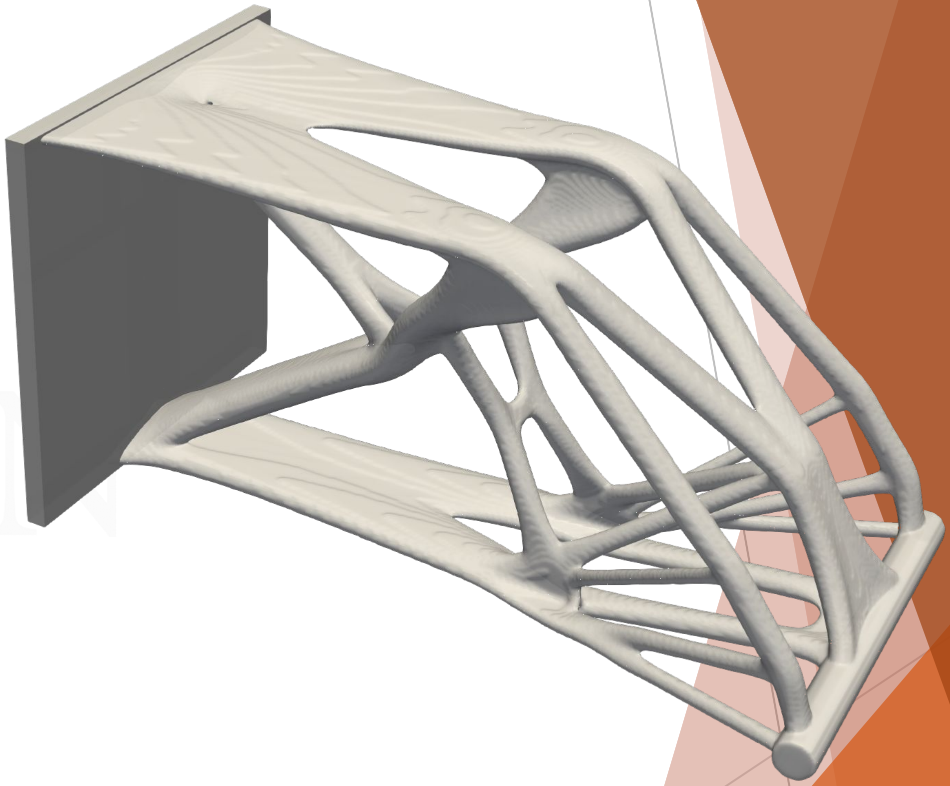
Mesh Independent Convergence





Cantilever – 3D



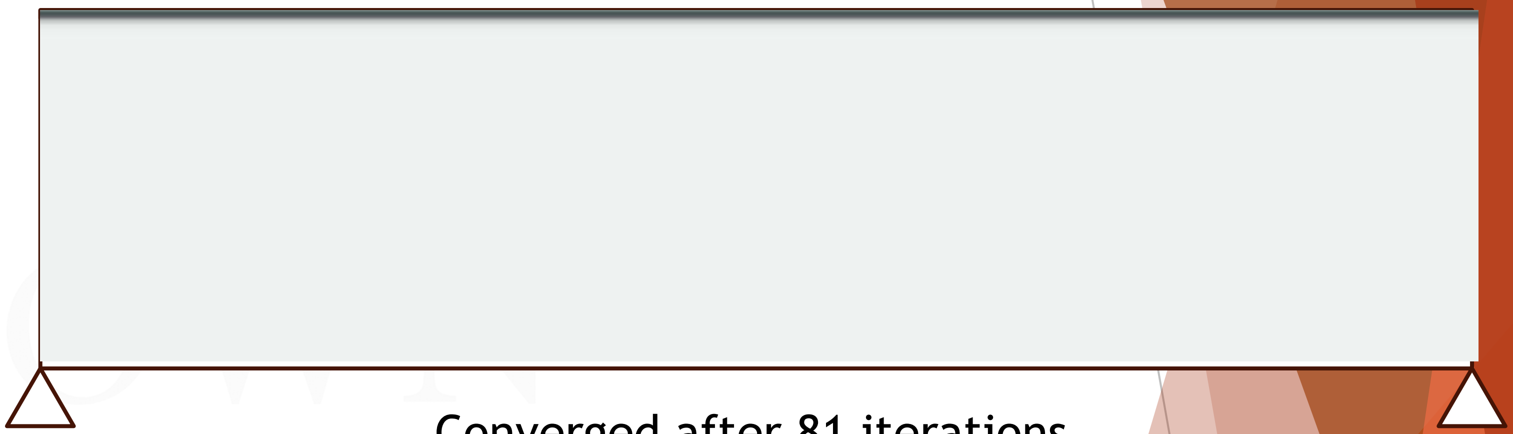


Converged after 44 iterations

Self-weight Compliance Minimization

- Self-weight bridge design

$$
\begin{aligned}
\min_{\rho} \quad & \mathbf{u}^T \mathbf{K}(\tilde{\boldsymbol{\rho}})\mathbf{u} \\
\text{subject to} \quad & \mathbf{K}(\tilde{\boldsymbol{\rho}})\mathbf{u} = \mathbf{f} + \mathbf{M}(\mathbf{g}\tilde{\boldsymbol{\rho}}) \\
& (\epsilon^2 \mathbf{A} + \mathbf{M})\tilde{\boldsymbol{\rho}} = \tilde{\mathbf{M}}\boldsymbol{\rho} \\
& 0 \leq \rho_i \leq 1, \\
& \mathbf{1}^T \mathbf{M}\boldsymbol{\rho} \geq \theta|\Omega|
\end{aligned}
$$

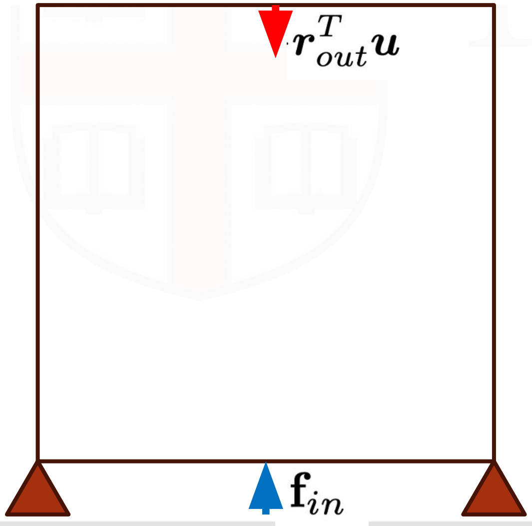


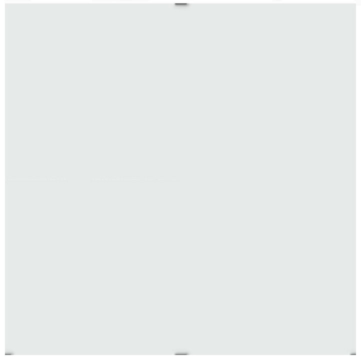
Converged after 81 iterations

Compliant Mechanism

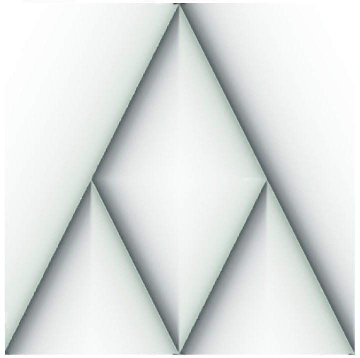
- Force Inverter

$$
\begin{aligned}
\min_{\rho} \quad & -\boldsymbol{r}_{out}^T \boldsymbol{u} \\
\text{subject to} \quad & \mathbf{K}(\tilde{\boldsymbol{\rho}})\mathbf{u} = \mathbf{f}_{in} \\
& (\epsilon^2 \mathbf{A} + \mathbf{M})\tilde{\boldsymbol{\rho}} = \tilde{\mathbf{M}}\boldsymbol{\rho} \\
& 0 \leq \rho_i \leq 1, \\
& \mathbf{1}^T \mathbf{M}\boldsymbol{\rho} \geq \theta|\Omega|
\end{aligned}
$$

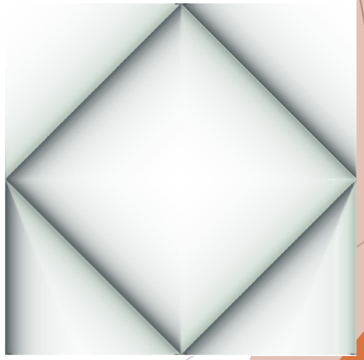




163 Iterations



126 Iterations



89 Iterations

Concluding Remarks

- We introduce the SiMPL method for topology optimization
  - The solution at each iteration is point-wise feasible
  - Update rule in the latent space is easy to implement
  - Computational cost at each iteration is equivalent to **gradient descent methods**
  - Backtracking line search results in fast and stable convergence
  - It outperforms OCM and MMA for all benchmark problems we tested
- A high-order interpolation in the latent space
  does NOT suffer from the oscillation in the primal space
  - Extend it to high-order scheme – Requires bound-preserving high-order filter solver
- Multi-material topology optimization
  - Multi-material constraints can be handled when we employ different entropy
- More general constraints
  - Stress constrained optimization, …

References

- B. Keith and T. M. Surowiec, Proximal Galerkin: A structure preserving finite element method for pointwise bound constraints, arXiv:2307.12444v5, 2024
- R. T. Rockafellar, Monotone operators and the proximal point algorithm, SIAM Journal on Control and Optimization, 14 (1976), pp. 877-898
- C. Blair, Problem complexity and method efficiency in optimization, SIAM Review, 27 (1985), pp. 264-265
- A. Beck and M. Teboulle, Mirror descent and nonlinear projected subgradient methods for convex optimization, Operations Research Letters, 31 (2003), pp. 167-175
- B. S. Lazrov and O. Sigmund, Filters in topology optimization based on Helmholtz-type differential equations, Numerical Methods in Engineering, 86 (2011), pp. 765-781
- M. P. Bendsøe, Optimization of Structural Topology, Shape, and Material, Springer Berlin Heidelberg, 1995
- K. Svanberg, The method of moving asymptotes – a new method for structural optimization, International Journal for Numerical Methods in Engineering, 24 (1987), pp. 359-373

SiMPL makes topology optimization SIMPLer!

Proximal Point Method

- Censor and Zenios 1992, Chen and Teboulle 1993

$$\min_{\boldsymbol{x}} \quad F(\boldsymbol{x})$$
$$\text{s.t.} \quad \boldsymbol{x} \in K$$

- The proximal point method is an iterative method

$$\boldsymbol{x}_{k+1} = \arg\min_{\boldsymbol{x}} \{F(\boldsymbol{x}) + \frac{1}{2\alpha_k}\|\boldsymbol{x} - \boldsymbol{x}_k\|^2\}$$

- We replace the Euclidean distance to a Bregman distance

$$\boldsymbol{x}_{k+1} = \arg\min_{\boldsymbol{x}} \{F(\boldsymbol{x}) + \frac{1}{\alpha_k} D_h(\boldsymbol{x}, \boldsymbol{x}_{k+1})\}$$

where the Bregman divergence is defined by

$$D_h(\boldsymbol{x}, \boldsymbol{y}) = h(\boldsymbol{x}) - h(\boldsymbol{y}) - \langle h'(\boldsymbol{y}), \boldsymbol{x} - \boldsymbol{y} \rangle$$

with a strictly convex and differentiable function h

Proximal Minimization Algorithm with *D*-Functions[1,2]

Y. CENSOR[3] AND S. A. ZENIOS[4]

Communicated by O. L. Mangasarian

Abstract. The original proximal minimization algorithm employs quadratic additive terms in the objectives of the subproblems. In this paper, we replace these quadratic additive terms by more general *D*-functions which resemble (but are not strictly) distance functions. We characterize the properties of such *D*-functions which, when used in the proximal minimization algorithm, preserve its overall convergence. The quadratic case as well as an entropy-oriented proximal minimization algorithm are obtained as special cases.

CONVERGENCE ANALYSIS OF A PROXIMAL-LIKE MINIMIZATION ALGORITHM USING BREGMAN FUNCTIONS*

GONG CHEN† AND MARC TEBOULLE†

Abstract. An alternative convergence proof of a proximal-like minimization algorithm using Bregman functions, recently proposed by Censor and Zenios, is presented. The analysis allows the establishment of a global convergence rate of the algorithm expressed in terms of function values.

Key words. Bregman functions, proximal methods, convex programming

AMS subject classification. 90C25

1. Introduction. Consider the convex optimization problem

(1) $$(P) \qquad \min\{f(x) : x \in \mathbb{R}^n\},$$

where $f : \mathbb{R}^n \mapsto (-\infty, +\infty]$ is a proper, lower semicontinuous convex function. One method of solving (P) is to regularize the objective function by using the proximal mapping as introduced by Moreau [12]. Given a real positive number λ, a proximal approximation of f is defined by

(2) $$f_\lambda(x) = \inf_u\{f(u) + 1/2\lambda\|x - u\|^2\}.$$

Proximal Point Method

- The proximal iterate

$$\boldsymbol{x}_{k+1} = \arg\ \min_{\boldsymbol{x}}\{F(\boldsymbol{x}) + \frac{1}{\alpha_k} D_h(\boldsymbol{x}, \boldsymbol{x}_{k+1})\}$$

can be found by solving the first-order optimality condition

$$\alpha \nabla_{\boldsymbol{x}} F(\boldsymbol{x}) + \nabla_{\boldsymbol{x}} h(\boldsymbol{x}) = \nabla_{\boldsymbol{x}} h(\boldsymbol{x}_k)$$

- When we consider a constrained optimization problem,
h is often selected so that $dom(h) = K$
and its gradient has closed form inverse

- Then we always have feasible iterates

- Converges provided with $\sum \alpha_k \to \infty$

Latent Variable Proximal Point Method

- Keith and Surowiec 2023

- If we define dual variable, $\boldsymbol{x}^* = \nabla_{\boldsymbol{x}} h(\boldsymbol{x})$, then

$$
\begin{aligned}
\alpha_k \nabla_{\boldsymbol{x}} F(\boldsymbol{x}_{k+1}) + \boldsymbol{x}^*_{k+1} &= \boldsymbol{x}^*_k \\
\boldsymbol{x}_{k+1} - \nabla_{\boldsymbol{x}} h^{-1}(\boldsymbol{x}^*_{k+1}) &= 0.
\end{aligned}
$$

- In a continuous setting, the dual variable has no crucial role

- However, in a discrete setting,

$$
\begin{aligned}
\alpha_k \nabla_{\boldsymbol{x}} F_h(\boldsymbol{x}_{h,k+1}) + \boldsymbol{x}^*_{h,k+1} &= \boldsymbol{x}^*_{h,k} \\
\boldsymbol{x}_{h,k+1} - \nabla_{\boldsymbol{x}} h^{-1}(\boldsymbol{x}^*_{h,k+1}) &= 0,
\end{aligned}
$$

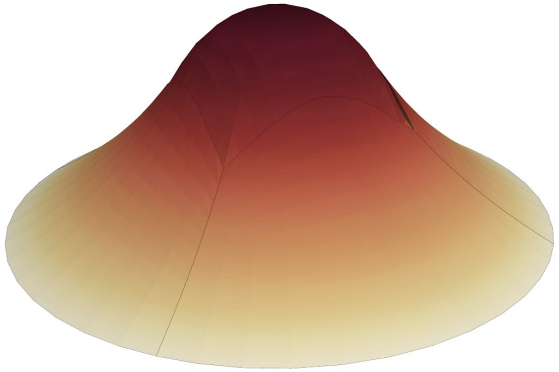
we have an additional representation of solution, $\boldsymbol{x} \approx \nabla_{\boldsymbol{x}} h^{-1}(\boldsymbol{x}^*_{h})$
This is feasible point-wisely

PROXIMAL GALERKIN: A STRUCTURE-PRESERVING FINITE ELEMENT METHOD FOR POINTWISE BOUND CONSTRAINTS

BRENDAN KEITH* AND THOMAS M. SUROWIEC†

Dedicated with respect and admiration to Leszek Demkowicz on the occasion of his 70th birthday anniversary.

Abstract. The proximal Galerkin finite element method is a high-order, low iteration complexity, nonlinear numerical method that preserves the geometric and algebraic structure of pointwise bound constraints in infinite-dimensional function spaces. This paper introduces the proximal Galerkin method and applies it to solve free boundary problems, enforce discrete maximum principles, and develop a scalable, mesh-independent algorithm for optimal design problems with pointwise bound constraints. This paper also provides a derivation of the latent variable proximal point (LVPP) algorithm, an unconditionally stable alternative to the interior point method. LVPP is an infinite-dimensional optimization algorithm that may be viewed as having an adaptive barrier function that is updated with a new informative prior at each (outer loop) optimization iteration. One of its main benefits is witnessed when analyzing the classical obstacle problem. Therein, we find that the original variational *inequality* can be replaced by a sequence of second-order partial differential *equations* (PDEs) that are readily discretized and solved with, *e.g.*, high-order finite elements. Throughout ... may be of independent interest. The ... *sson equation*; (2) an algebraic/geo ... tizations and certain infinite-dimen ... gorithm for two-field, density-based ... Galerkin methodology combines ide ... ebra, and differential geometry and ... as within variational and numerica ... tions of our methods to facilitate r



Latent Variable Mirror Descent

- The first-order approximation of objective function

$$\boldsymbol{x}_{k+1} = \arg\min_{\boldsymbol{x}} \{F(\boldsymbol{x}_k) + (\nabla_{\boldsymbol{x}} F(\boldsymbol{x}_k), \boldsymbol{x} - \boldsymbol{x}_k) + \frac{1}{\alpha_k} D_h(\boldsymbol{x}, \boldsymbol{x}_{k+1})\}$$
$$= \arg\min_{\boldsymbol{x}} \{(\nabla_{\boldsymbol{x}} F(\boldsymbol{x}_k), \boldsymbol{x}) + \frac{1}{\alpha_k} D_h(\boldsymbol{x}, \boldsymbol{x}_{k+1})\}$$

- This method is called mirror descent, and it has explicit form

$$\boldsymbol{x}_{k+1} = \nabla h_{\boldsymbol{x}}^{-1}(\nabla_{\boldsymbol{x}} h(\boldsymbol{x}_k) - \alpha_k \nabla_{\boldsymbol{x}} F(\boldsymbol{x}_k))$$

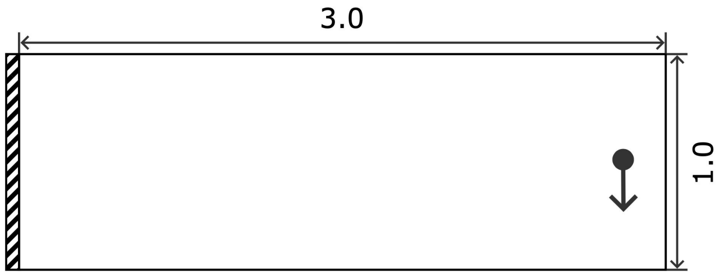
- In terms of the latent variable, $\boldsymbol{x}^* = \nabla_{\boldsymbol{x}} h(\boldsymbol{x})$

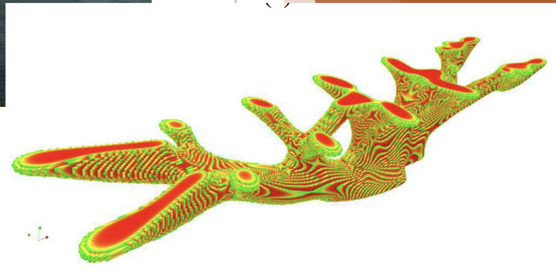
$$\boldsymbol{x}^*_{k+1} = \boldsymbol{x}^*_k - \alpha_k \nabla_{\boldsymbol{x}} F(\boldsymbol{x}_k)$$

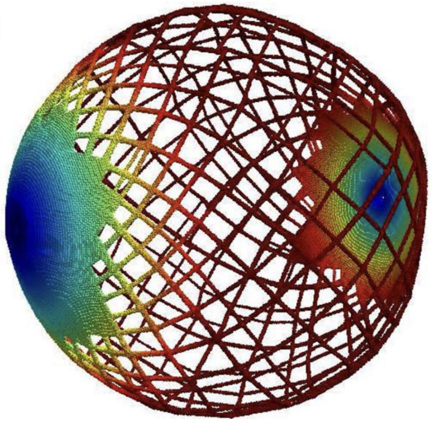
- What is Topology Optimization
- Projected Mirror Descent with Latent Variable
- Adaptive Step Size - Guided Initial Guess

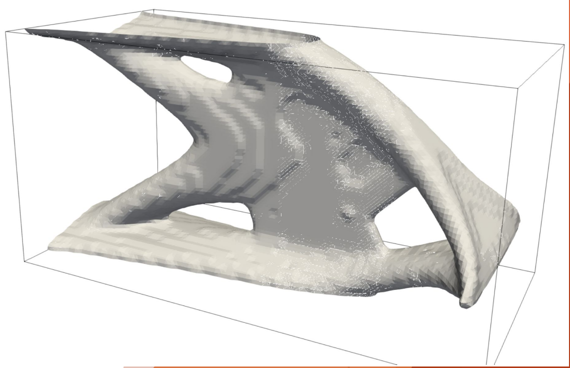
Topology Optimization

Topology Optimization





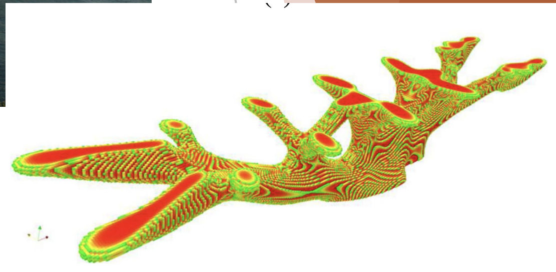


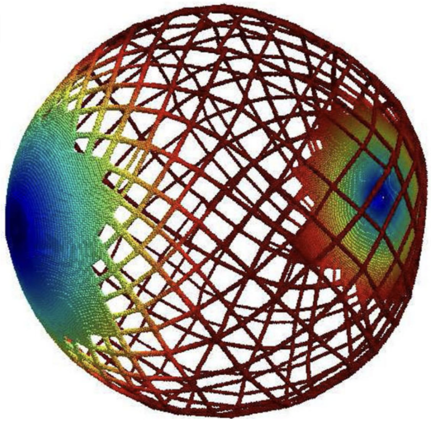


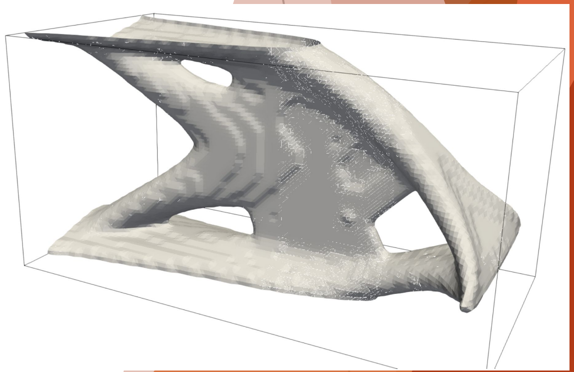
Topology Optimization

$$
\begin{aligned}
\min_{\rho \in L^\infty(\Omega)} \quad & (\boldsymbol{f}, \boldsymbol{u}) \\
\text{s.t.} \quad & \nabla \cdot (r(\tilde{\rho})\mathcal{C} : \varepsilon(\boldsymbol{u})) = \boldsymbol{f} \quad \text{in } \Omega \; + \; \text{B.Cs} \\
& -\epsilon^2 \Delta \tilde{\rho} + \tilde{\rho} = \rho \quad \text{in } \Omega, \\
& \partial_n \tilde{\rho} = 0 \quad \text{on } \partial\Omega, \\
& \int_\Omega \rho = \theta|\Omega|, \\
& 0 < \rho < 1.
\end{aligned}
$$

where $r(\tilde{\rho}) = \rho_0 + (1 - \rho_0)\tilde{\rho}^{p}$





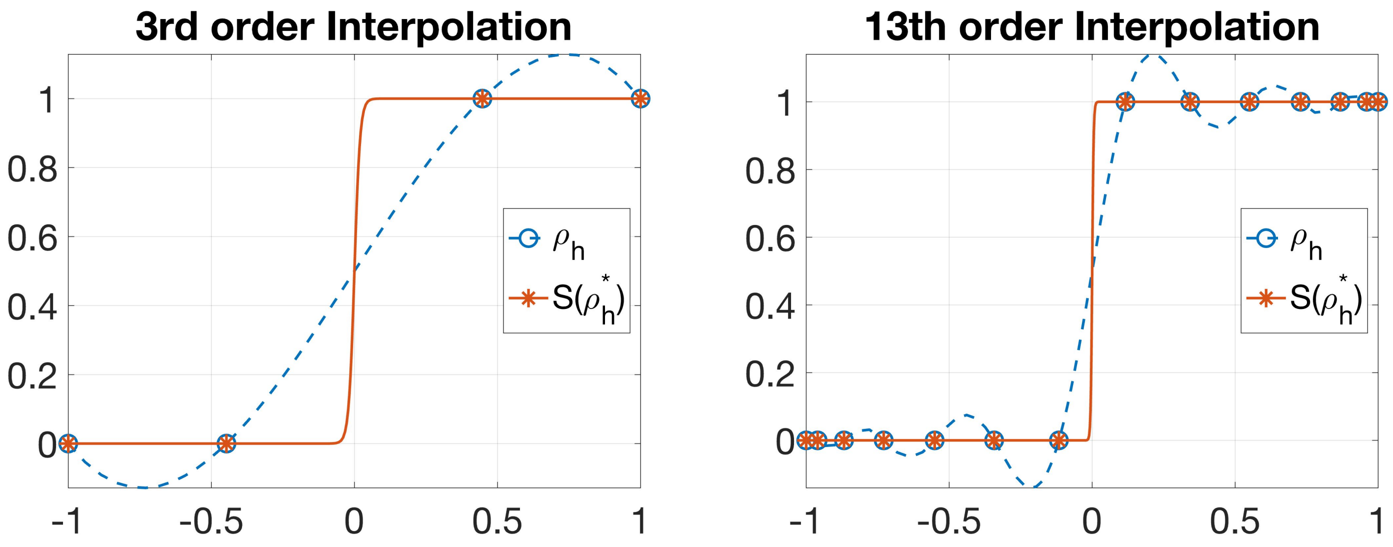


Topology Optimization

- We choose $h(\rho) = \int_\Omega \rho \ln \rho + (1-\rho)\ln(1-\rho)$

$$S^{-1}(\rho) := \nabla_\rho h(\rho) \qquad = \ln \rho - \ln(1-\rho) = \ln \frac{\rho}{1-\rho}$$

$$S(\rho^*) := \nabla_\rho h^{-1}(\rho^*) \qquad = \frac{1}{1+\exp(-\rho^*)} \in (0,1)$$



Projected Mirror Descent Method

Algorithm 2 Projected Mirror Descent

1: Given dual density $\rho_k^* \in Q_h^*$ and step size α_k.

2: Solve for filtered density $\tilde{\rho}_k \in \tilde{Q}_h$,

$$\epsilon^2(\nabla\tilde{\rho}_k, \nabla\tilde{q}) + (\tilde{\rho}_k, \tilde{q}) = (S(\rho_k^*), q) \quad \forall \tilde{q} \in \tilde{Q}_h.$$

3: Solve for displacement $\boldsymbol{u}_k \in V_h$

$$(r(\tilde{\rho}_k)\mathcal{C} : \varepsilon(\boldsymbol{u}_h), \nabla\boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v}) \quad \forall \boldsymbol{v} \in V_h.$$

4: Compute gradient $G_k = \Pi_h \tilde{w}_h \in Q_h^*$ where $\tilde{w}_h \in \tilde{Q}_h$ solves

$$\epsilon^2(\nabla\tilde{w}_k, \nabla\tilde{q}) + (\tilde{w}_k, \tilde{q}) = -(r'(\tilde{\rho}_k)(\mathcal{C} : \varepsilon(\boldsymbol{u}_k)) : \varepsilon(\boldsymbol{u}_k), \tilde{q}) \quad \forall \tilde{q} \in \tilde{Q}_h.$$

5: Set $\rho_{k+1}^* = \rho_k^* - \alpha_k G_k + \mu$ where $\mu \in \mathbb{R}$ solves

$$\int_\Omega S(\rho_k^* - \alpha_k G_k + \mu) = \theta|\Omega|$$

Adaptive Step Size

- The convergence of Mirror Descent can be proved

$$\alpha_k^{-1} \geq \gamma^{-1} := \sup_{\boldsymbol{x},\boldsymbol{y}\in K} \frac{(\nabla F(\boldsymbol{x}) - \nabla F(\boldsymbol{y}), \boldsymbol{x} - \boldsymbol{y})}{(\nabla h(\boldsymbol{x}) - \nabla h(\boldsymbol{y}), \boldsymbol{x} - \boldsymbol{y})}$$

$$= \sup_{\boldsymbol{x},\boldsymbol{y}\in K} \frac{(\nabla F(\boldsymbol{x}) - \nabla F(\boldsymbol{y}), \boldsymbol{x} - \boldsymbol{y})}{(\boldsymbol{x}^* - \boldsymbol{y}^*, \boldsymbol{x} - \boldsymbol{y})}$$

- This motivates the choice of step size

$$\alpha_k := \frac{(\rho_{h,k}^* - \rho_{h,k-1}^*, \rho_{h,k} - \rho_{h,k-1})}{(\nabla F(\rho_{h,k}) - \nabla F(\rho_{h,k-1}), \rho_{h,k} - \rho_{h,k-1})}$$

- Armijo condition is used to have monotone decreasing objective

$$F(\rho_{h,k+1}) \leq F(\rho_{h,k}) - c_1(\nabla F(\rho_{h,k}), \rho_{h,k+1} - \rho_{h,k})$$

Main Results

1. $h : L^{\infty}_{(0,1)}(\Omega) \to \mathbb{R}$ is strictly convex and differentiable. Also, we can find primal representation, $\nabla h \in L^{\infty}(\Omega)$ by
$$(\nabla_{\rho} h(\rho), q) = (\ln \frac{\rho}{1-\rho}, q) \quad \forall q \in L^1(\Omega).$$
2. F is differentiable with respect to ρ. Its gradient $\nabla_{\rho} F(\rho) \in L^{\infty}(\Omega)$ can be represented by
$$(\nabla_{\rho} F(\rho), q) = (\tilde{w}, q) \quad \forall q \in L^1(\Omega).$$
3. The step size α_k satisfies the Armijo condition in finite iteration with
$$\alpha_k \geq \frac{1-c_1}{L}$$
where L is Lipschitz constant of $\nabla_{\rho} F$.
4. $\{F(\rho_k) = F(S(\rho_k^*))\}$ is a monotone decreasing sequence, and has a convergent subsequence
$$\rho_{\ell_k} \rightharpoonup \rho^*$$
and ρ^* satisfies the KKT condition.

Numerical Results

- MBB Beam
- Self-weight Compliance Minimization
- Compliant Mechanism

Discrete Spaces and Parameters

- Discrete Spaces

$$
\begin{aligned}
Q_h^* &= \{\rho^* \in L^2(\Omega) : \rho^*|_T \in \mathbb{P}_0(T)\ \forall T \in \mathcal{T}_h\},\\
\tilde{Q}_h &= \{\tilde{\rho} \in H^1(\Omega) : \tilde{\rho}|_T \in \mathbb{P}_1(T)\ \forall T \in \mathcal{T}_h\},\\
V_h &= \{\boldsymbol{u} \in [H^1(\Omega)]^2 : \boldsymbol{u}|_T \in [\mathbb{P}_1(T)]^2\ \forall T \in \mathcal{T}_h\}
\end{aligned}
$$

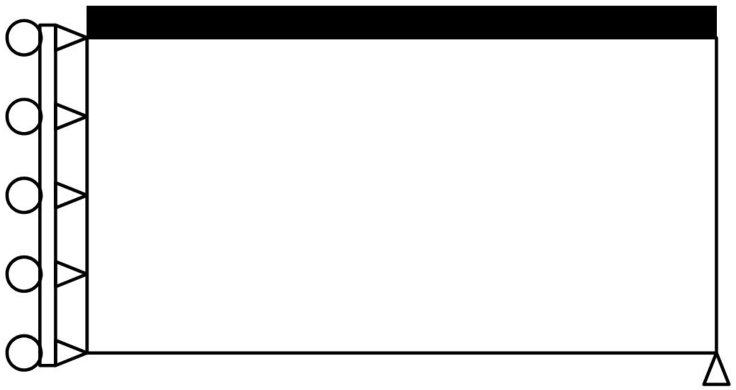
- Parameters

$$
\begin{aligned}
\epsilon &= 0.05/(2\sqrt{3}),\\
r(\tilde{\rho}) &= \rho_0 + (1-\rho_0)\tilde{\rho}^3\\
\rho_0 &= 10^{-6},\\
c_1 &= 10^{-4}.
\end{aligned}
$$

- MFEM: Open source C++ FEM Library @ LLNL

Self-weight Compliance Minimization

$$
\begin{aligned}
\min_{\rho \in L^\infty(\Omega)} \quad & (\boldsymbol{g}\rho, \boldsymbol{u}) \\
\text{s.t.} \quad & \nabla \cdot (\rho \mathcal{C} : \varepsilon(\boldsymbol{u})) = \boldsymbol{f} \quad \text{in } \Omega \ + \ \text{B.Cs} \\
& \int_\Omega \rho = \theta |\Omega|, \\
& 0 < \rho < 1.
\end{aligned}
$$



Compliant Mechanism

$$\min_{\rho \in L^\infty(\Omega)} \quad u_{out}$$

$$\text{s.t.} \quad \nabla \cdot (\rho \mathcal{C} : \varepsilon(\boldsymbol{u})) = \boldsymbol{f}_{in} \quad \text{in } \Omega \ + \ \text{B.Cs}$$

$$\int_\Omega \rho = \theta |\Omega|,$$

$$0 < \rho < 1.$$

Conclusion

- We derived Latent Variable Projected Mirror Descent Method
- The discrete density satisfies the bound constraint point-wisely even when high-order approximation is used
- Adaptive step size with a heuristic initial guess outperform traditional methods
- Efficiency and robustness of the proposed algorithm has been shown numerically
- High-order approximation also can be employed while maintaining bound-preserving property