Characteristic Bending

A Robust Advection Scheme for Incompressible Flows

Matthew Blomquist

Ph.D. Candidate, Applied Mathematics

University of California, Merced

Characteristic Bending

Big Picture

Question

If we know that our velocity field is (or should be) incompressible, can we use this information to design a robust advection scheme?

Spatial Interpolation Error

Time Integration Error

Velocity Error (Inexact Projection)

Characteristic Bending

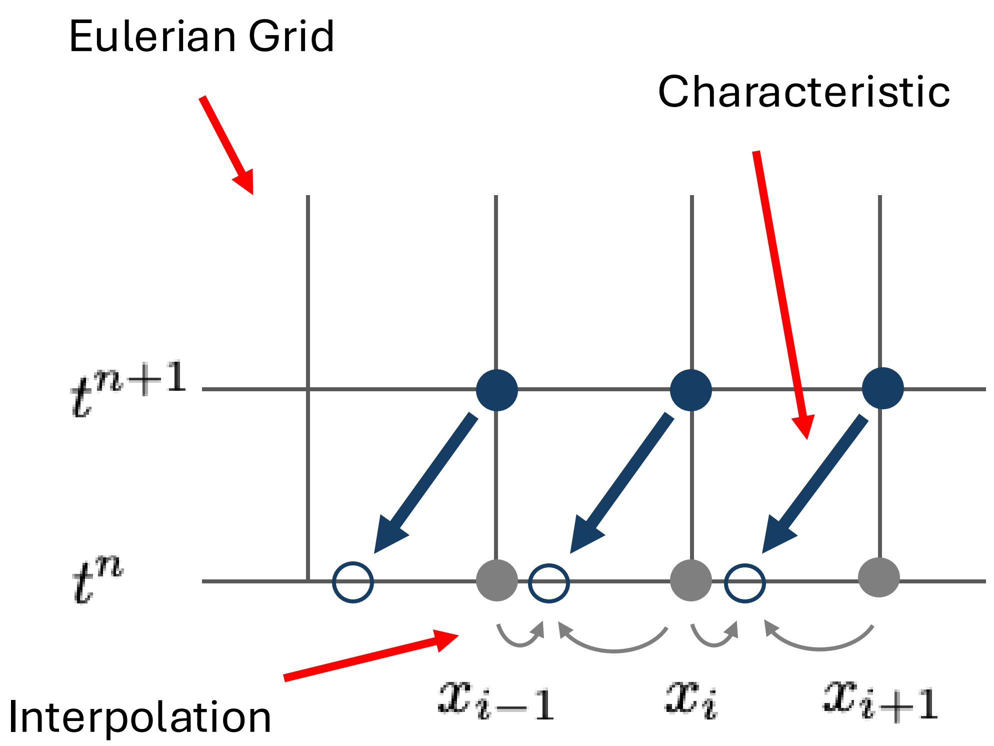
Semi-Lagrangian Method

Eulerian Frame

$$\frac{\partial \phi}{\partial t} + \mathbf{u} \cdot \nabla \phi = 0$$

Lagrangian Frame

$$\frac{Dx}{Dt} = \mathbf{u} \qquad \frac{D\phi}{Dt} = 0$$

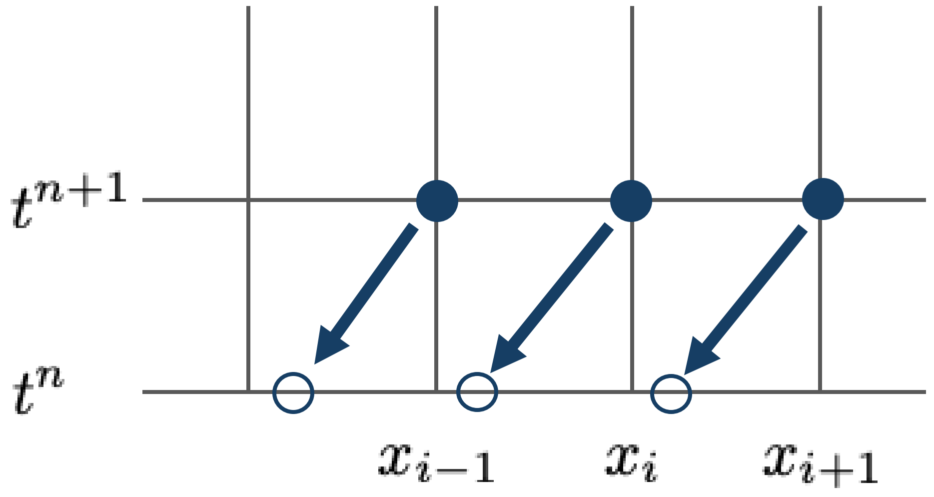


Characteristic Bending
Semi-Lagrangian Method

- Pros:
  - Unconditionally stable for linear advection (no CFL restriction).
  - Very good at resolving fronts, sub-grid features.

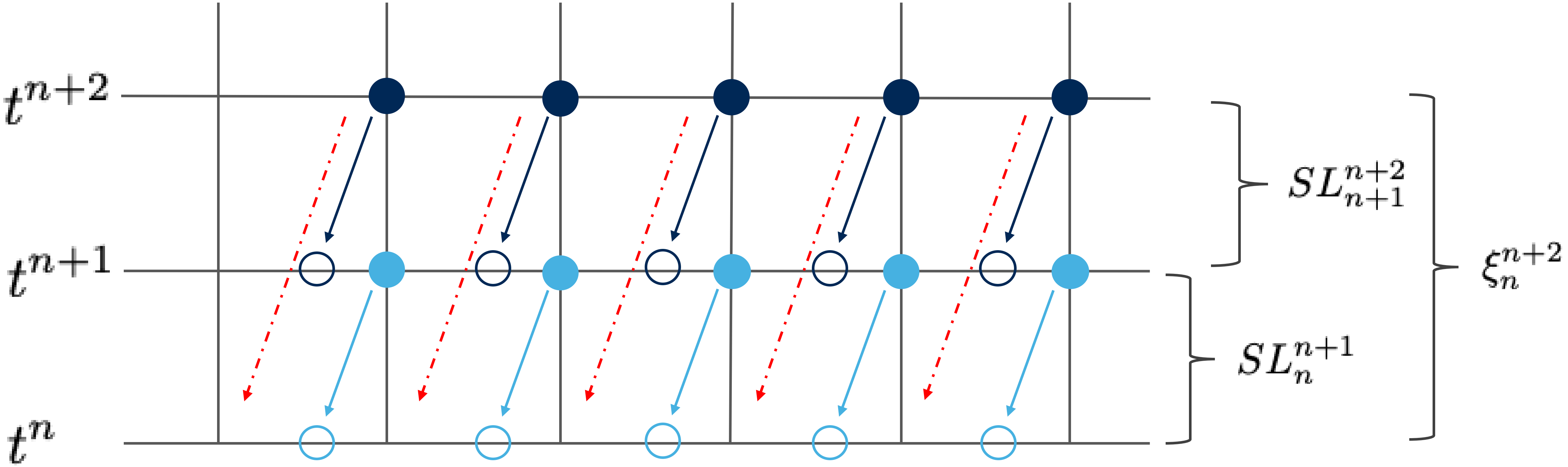
- Cons:
  - Single step method.
  - No inherent mechanism for preserving incompressibility, conservation, etc.

Semi-Lagrangian



Characteristic Bending

Characteristic Reconstruction



Characteristic Bending

Reference Map Advection

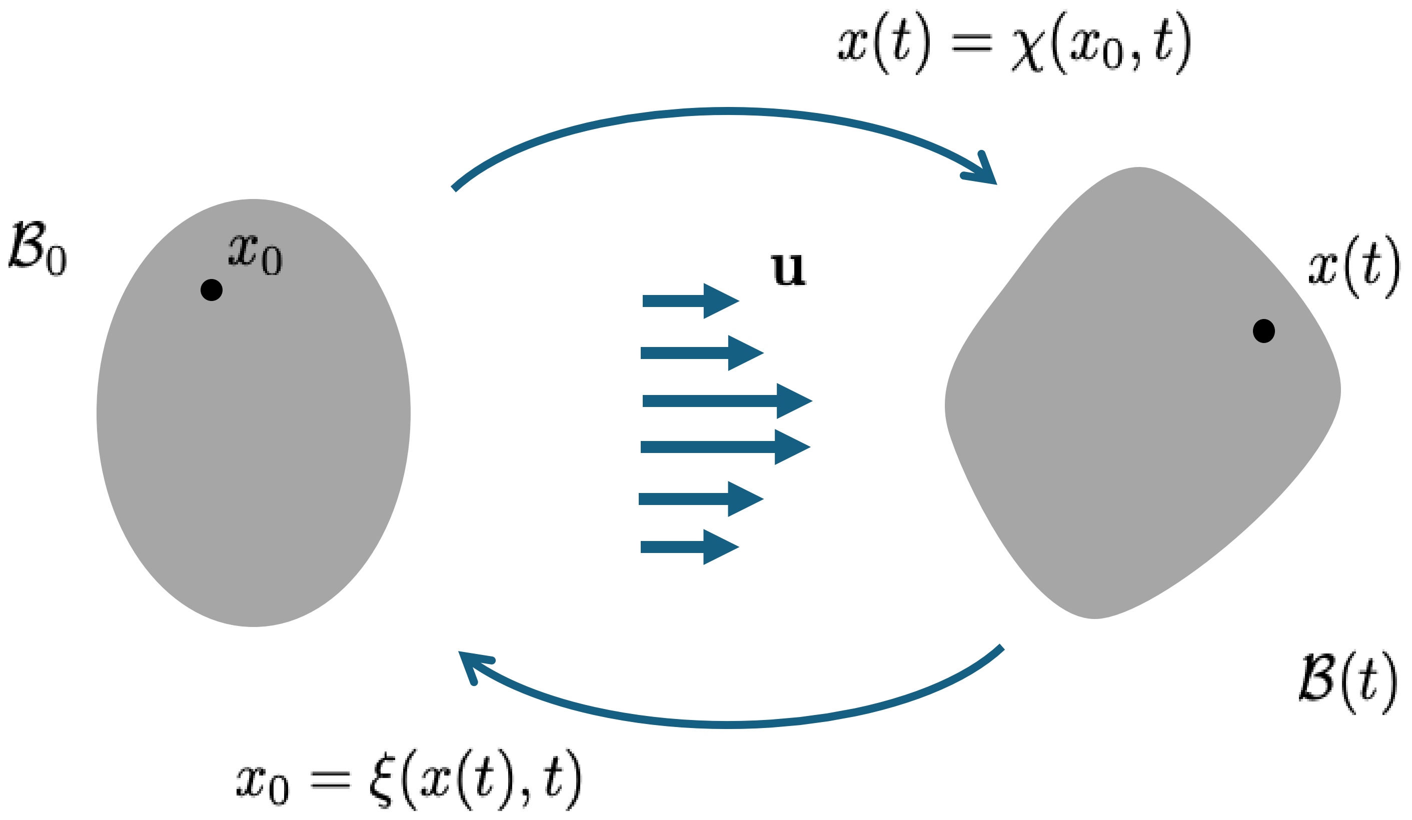
$$\frac{\partial \phi}{\partial t} + \mathbf{u} \cdot \nabla \phi = 0$$

Advection

$$\frac{\partial \xi}{\partial t} + \mathbf{u} \cdot \nabla \xi = 0$$

Reconstruction

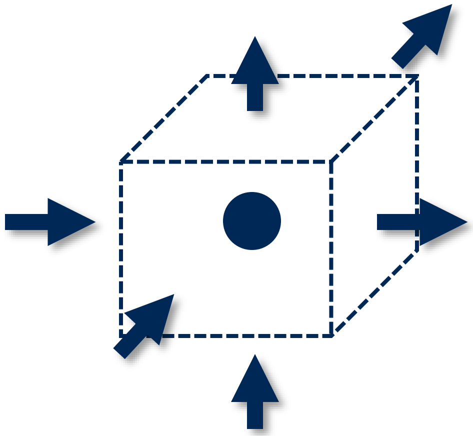
$$\phi(x,t) = \phi(\xi(x(t),0))$$



Characteristic Bending

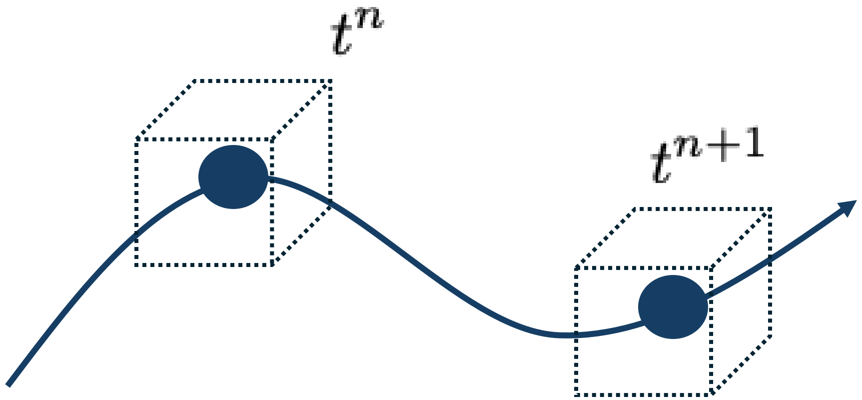
Incompressibility vs. Volume Preservation

Eulerian Frame



$$\nabla \cdot \mathbf{u} = 0$$

Lagrangian Frame

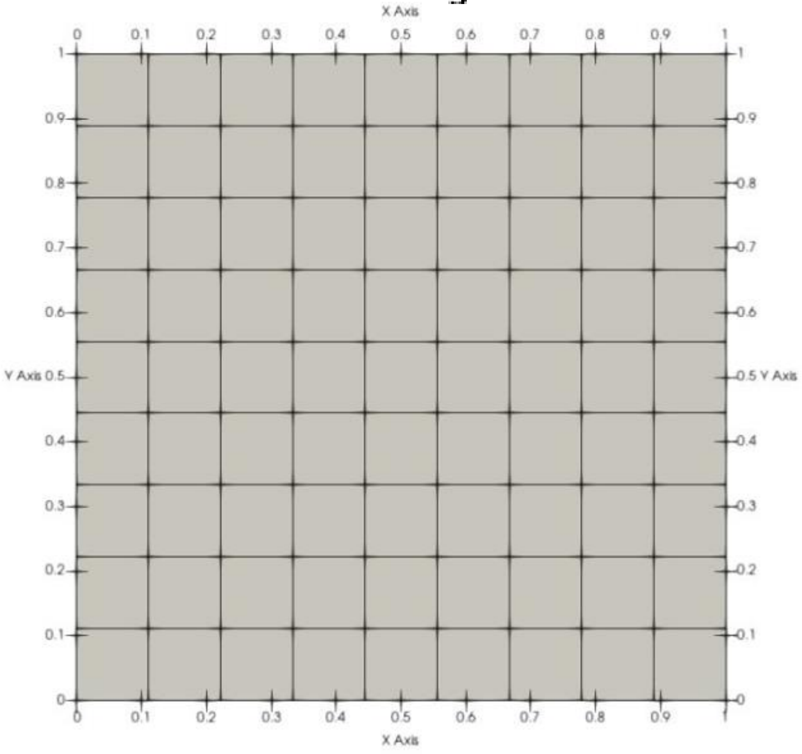


$$V^n = V^{n+1}$$

Characteristic Bending

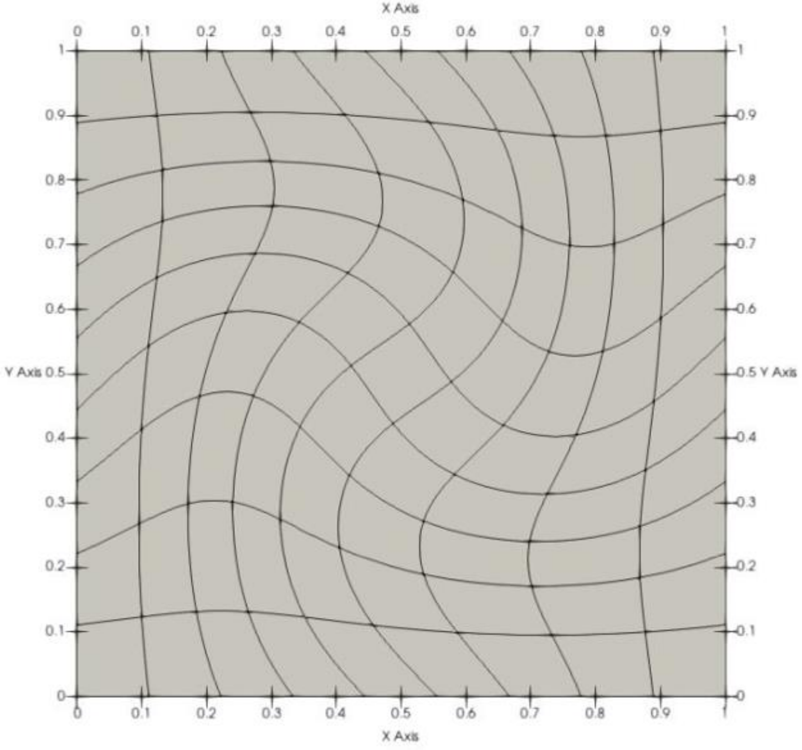
Deformation of the Space

$$\nabla\xi^0$$



$$det(\nabla\xi) = 1$$

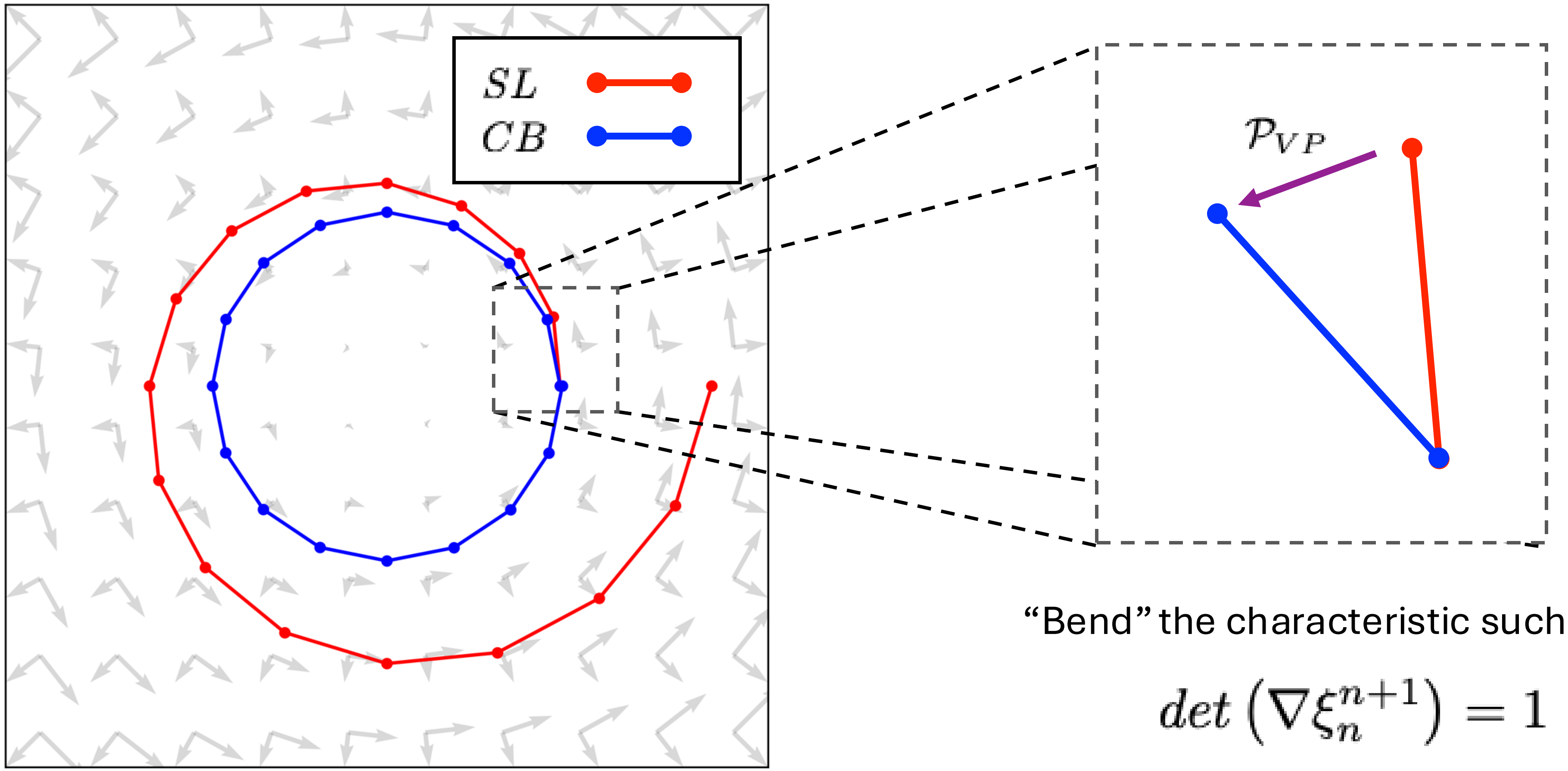
$$\nabla\xi^n$$



$$det(\nabla\xi) \neq 1$$

Characteristic Bending

Bending the Characteristics



"Bend" the characteristic such that:

$$det\left(\nabla \xi_n^{n+1}\right) = 1$$

Characteristic Bending

Formal Algorithm

1. Initialize and advect a Reference Map

$$\frac{\partial \xi}{\partial t} + \mathbf{u} \cdot \nabla \xi = 0$$

2. Project onto the Volume Preserving Space

Solve for the adjoint λ:

$$-\Delta \lambda = 1 - \det\left(\nabla \xi^*\right) \quad \forall \mathbf{x} \in \Omega$$

$$\lambda = 0 \qquad \forall x \in \partial\Omega$$

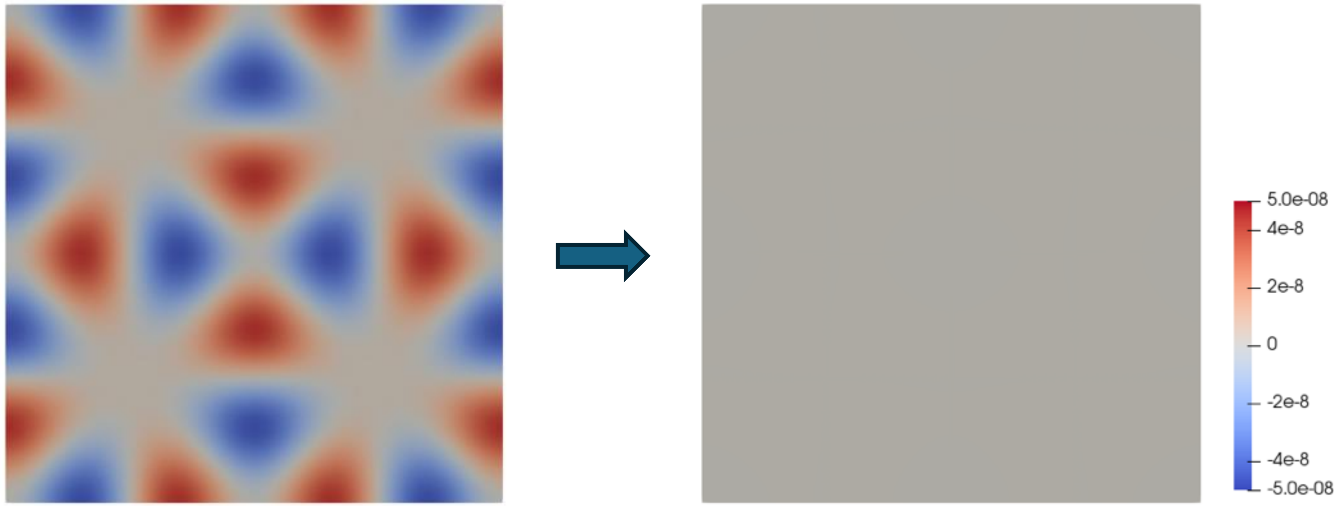
Compute the correction

$$\gamma^{-1}(x) = x - \nabla \lambda$$

3. Reconstruction

$$\phi^{n+1}(x) = \phi^n\left(\xi^*\left(\gamma^{-1}(x)\right)\right)$$

$$1 - \det\left(\nabla \xi^*(x)\right) \qquad 1 - \det\left(\nabla \xi^*\left(\gamma^{-1}(x)\right)\right)$$

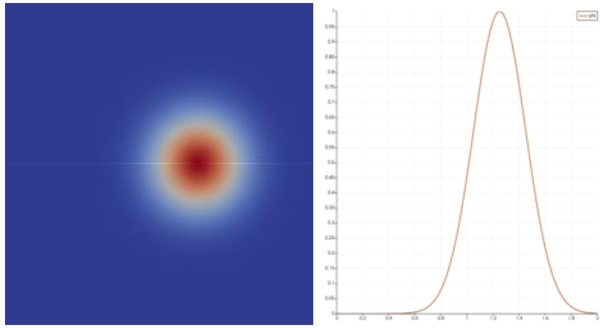


Intermediate Map   Corrected Map

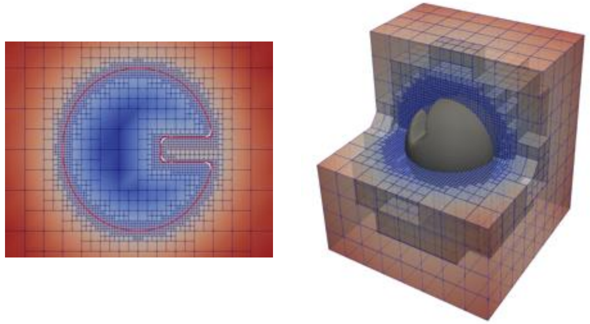
Characteristic Bending

Examples

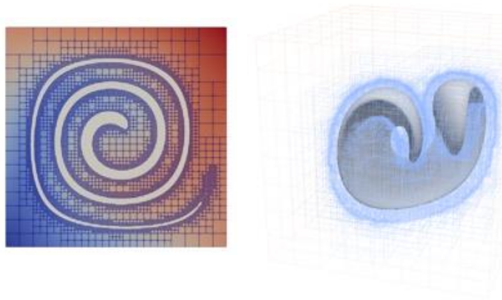
Gaussian Advection



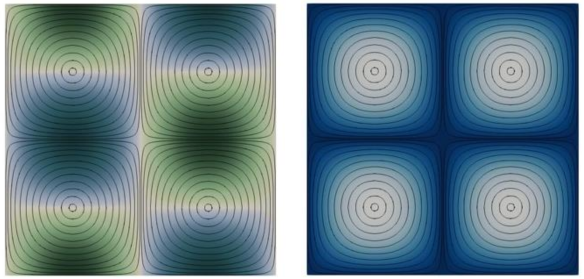
Solid Body Rotation



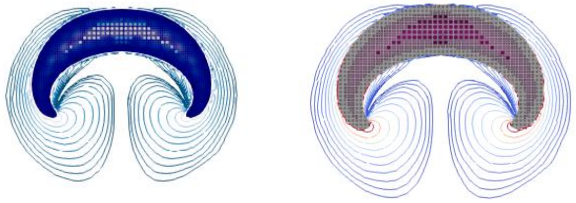
Deforming Flow



Incompressible Euler



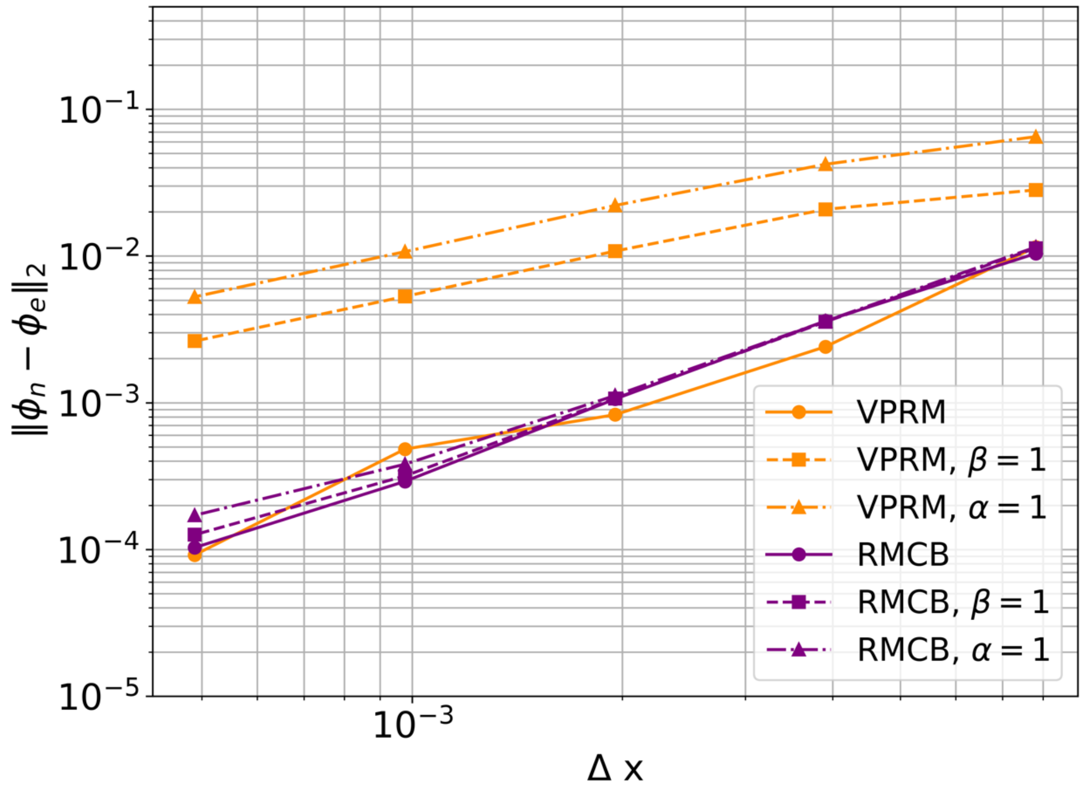
Multiphase Flows



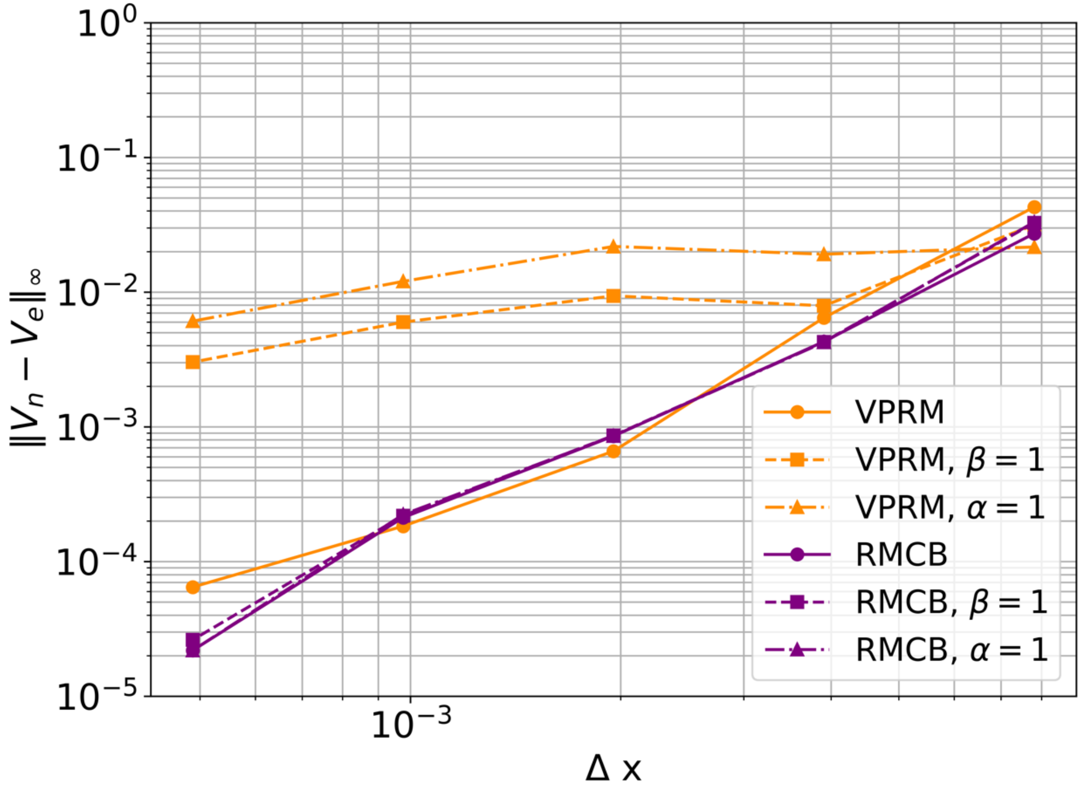
Characteristic Bending

Deforming Flow Results

Interface Error



Volume Error



Characteristic Bending

Non-linear Advection

Incompressible Euler

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p$$

$$\mathbf{u} = \begin{cases} \sin(x)\cos(y) \\ -\cos(x)\sin(y) \end{cases}$$

$$\nabla p = \begin{cases} \sin(x)\cos(x) \\ \sin(y)\cos(y) \end{cases}$$

$$\frac{Dx}{Dt} = \mathbf{u} \longrightarrow \text{Same}$$

$$\frac{D\mathbf{u}}{Dt} = -\nabla p$$

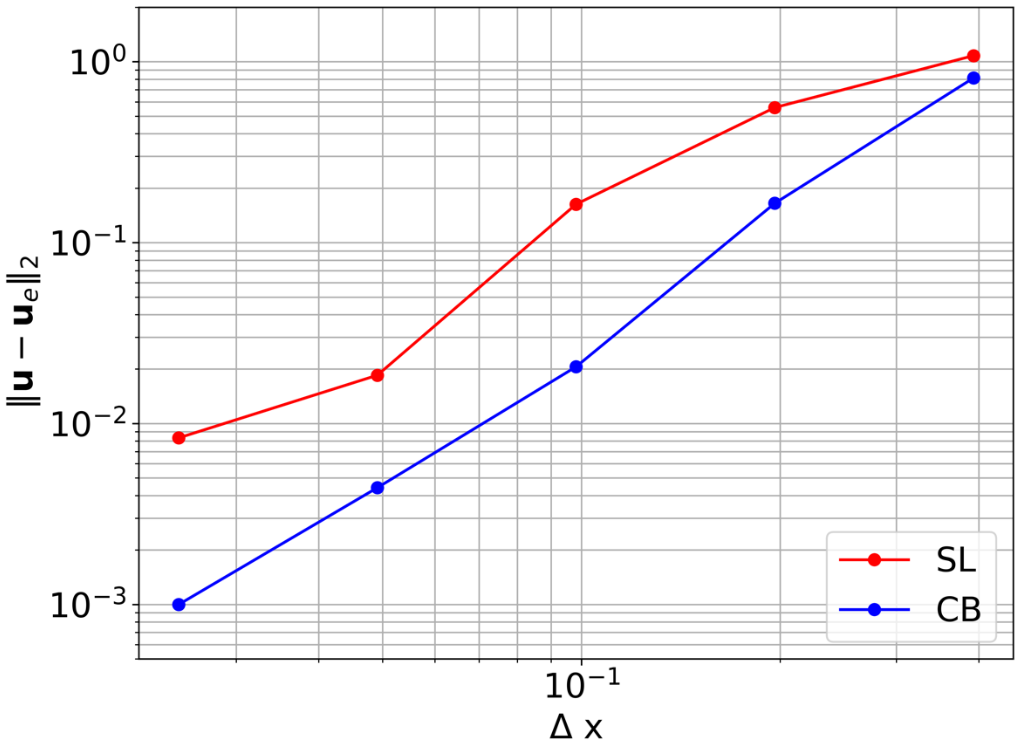
$$\mathbf{u}\left(x, t^{n+1}\right) = \mathbf{u}\left(x_d, t^n\right) - \int_{t^n}^{t^{n+1}} \nabla p\left(x(\tau), \tau\right) d\tau$$

New

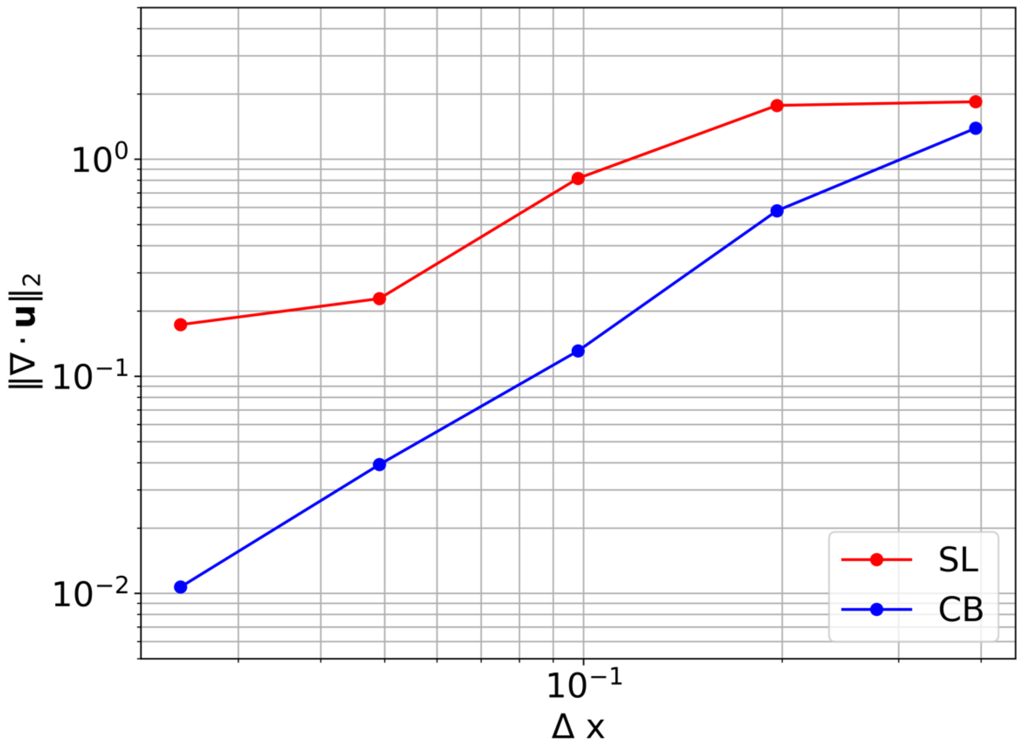
Characteristic Bending

Deviation from the Divergence Free Space

Velocity Error



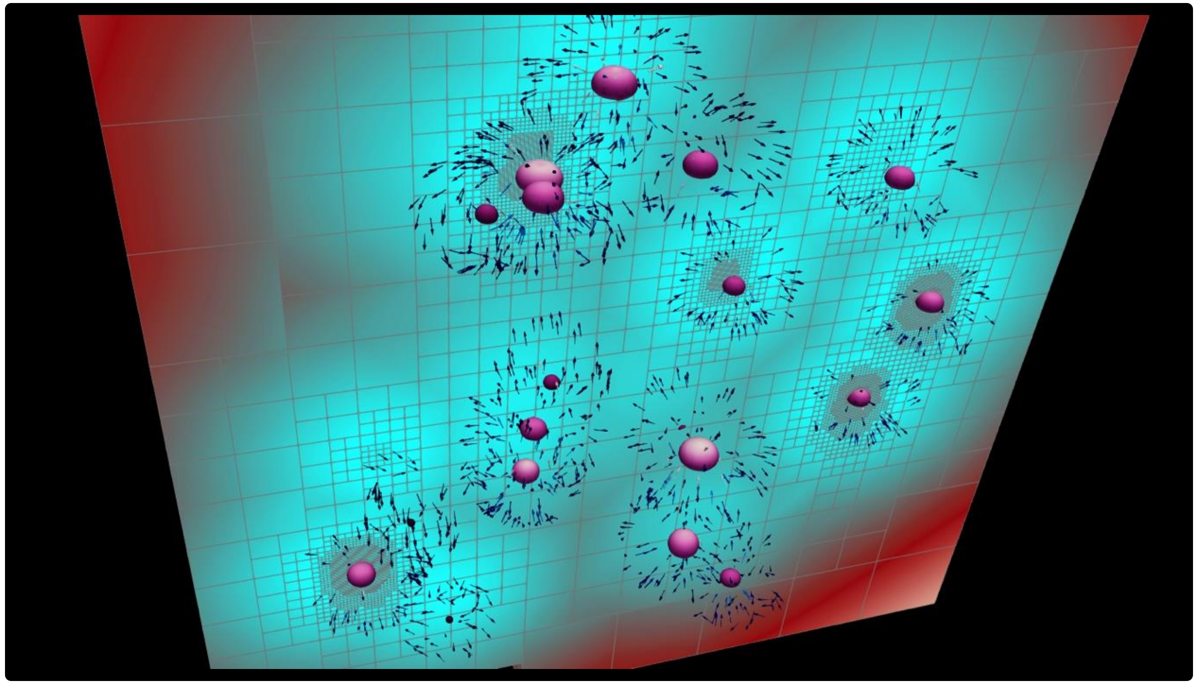
Divergence Error



Characteristic Bending

Summary and Next Steps

1. Characteristic Bending (CB) is a robust method for advection under incompressible flows.
2. CB filters compressible modes and preserves divergence-free fields.
3. CB is more expensive than foundational methods and no benefit for exact div-free fields.



Papers

- M. Blomquist, M. Theillard, "Semi-Lagrangian characteristic reconstruction and projection for transport under incompressible fields", *In preparation*.
- A. Binswanger*, M. Blomquist*, S. R. West, M. Theillard, "Sharp Collocated Projection Method for Immiscible Two-Phase Flows", arXiv preprint arXiv:2508:11107 (2025)
- M. Blomquist, S. R. West, A. Binswanger, M. Theillard, "Stable nodal projection method on octree grids", Journal of Computational Physics 499, 112695 (2024)

This material is based on work supported by the National Science Foundation under Grant No. DMS-1840265