A Stable FEM Framework for Coupled PDE–ODE Bioheat Models with Nonlinear Boundary Conditions.

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Motivation and Model: Classic Pennes Model

Pennes (Bioheat Transfer) Equation

$$c\rho u_t - \nabla \cdot (k \nabla u) + \mathbf{c_b \rho_b \omega_b (u - u_b)} = f,$$

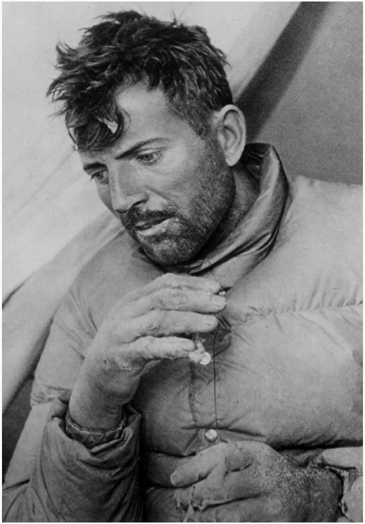
u : tissue temperature	u_b : blood temperature
ρ : tissue density	ρ_b : blood density
c : tissue specific heat	c_b : blood specific heat
k : tissue conductivity	ω_b : blood vol. flow rate

f: source (e.g. metabolism)

Blood–Tissue Energy Exchange

Units of $\mathbf{c_b \rho_b \omega_b}$ are [W m^{-3} $^\circ$C^{-1}].

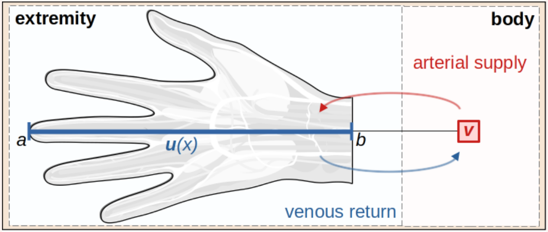
Interpret as *energy exchanged* per unit volume per degree difference in u and u_b.



Maurice Herzog, 1950, having lost his gloves leading the first successful climb of Annapurna.

Reference: Pennes (1948); *Analysis of Tissue and Arterial Blood Temperatures*

Motivation and Model



Bioheat Transfer Equation

$$cu_t - \nabla \cdot (k \nabla u) + \boldsymbol{R(u, v)} = f, \qquad (1a)$$

$$-k \nabla u \cdot \nu = \boldsymbol{\alpha(u, v)} \qquad (1b)$$

$$u(x, 0) = u_{\text{init}}(x) \qquad (1c)$$

Body Temperature

$$\kappa \frac{dv}{dt} + \boldsymbol{S(v, \langle u \rangle_\Omega)} = g \qquad (2a)$$

$$v(0) = v_{\text{init}} \qquad (2b)$$

Define $\langle u(t) \rangle_\Omega := |\Omega|^{-1} \int_\Omega u(\cdot, t)\, dx$

Assumptions

(a) $\boldsymbol{R}$, $\boldsymbol{S}$, $\boldsymbol{\alpha}$ are Lipschitz with constant L.

(b) $\alpha \in C^4(\Omega)$ and has derivatives bounded by L.

(c) $\kappa > 0$ and $c, k \in L^\infty(\Omega)$ with $0 < k_{\min} \leq k(x) \leq k_{\max} < \infty$ for a.e. $x \in \Omega$; similar for c.

(d) $f \in L^2(0, T; L^2(\Omega))$ and $g \in L^\infty(0, T)$

Linear Model: Coefficients

Constant Coefficient (Linear) Model

$$R(u,v) = \boldsymbol{A}(u - v)$$
$$S(u,v) = \boldsymbol{B}(v - \langle u \rangle_\Omega)$$

$$\alpha(u,v) = \begin{cases} C_{\text{air}}(u - u_{\text{air}}), & u \in \partial\Omega_{\text{air}}, \\ C_{\text{wrist}}(u - v), & u \in \partial\Omega_{\text{wrist}}. \end{cases}$$

	Cof.	Value	Source	Units	Motivation
Empirical	c	3.5×10^3	[1]	[J kg^{-1} $^\circ$C^{-1}]	heat capacity, muscle
	c_b	3617	[1]	[J kg^{-1} $^\circ$C^{-1}]	heat capacity, blood
	k	0.42	[1]	[W m^{-1} $^\circ$C^{-1}]	thermal conductivity, muscle
	ρ_b	1050	[1]	[kg m^{-3}]	density, blood
	ω_b	1.1×10^{-3}	[2]	[L L^{-1} s^{-1}]	normothermic hand blood flow rate
	$\boldsymbol{A}$	$\boldsymbol{c_b \rho_b \omega_b}$	[3]	[W m^{-3} $^\circ$C^{-1}]	coefficient from Pennes equation
Semi-Empirical	u_0	34	[2]	[$^\circ$C]	approximate mean skin temperature
	v_0	37	[2]	[$^\circ$C]	normothermic deep body temperature
	f	0	[2]	[W m^{-3}]	hands produce little internal heat
	g	700	[4]	[W m^{-3}]	maximal shivering metabolic heat generation rate
	$\boldsymbol{C_{\text{air}}}$	**136**	[2]	[W m^{-2} $^\circ$C^{-1}]	energy dissipation coefficient from
	u_{air}	-40	[5]	[$^\circ$C]	external temp. where exposed skin freezes in seconds
Heuristic	u_1	10	[5]	[$^\circ$C]	little or no local blood flow
	u_2	32		[$^\circ$C]	normothermic local blood flow
	v_1	28		[$^\circ$C]	cessation of extremity blood flow
	v_2	37		[$^\circ$C]	normothermic extremity blood flow
	$\boldsymbol{B}$	**0.07 $\boldsymbol{A}$**		[W m^{-3} $^\circ$C^{-1}]	*ad hoc*
	$\boldsymbol{C_{\text{wrist}}}$	**100**		[W m^{-2} $^\circ$C^{-1}]	*ad hoc*

Table 1: Coefficients used in numerical experiments.

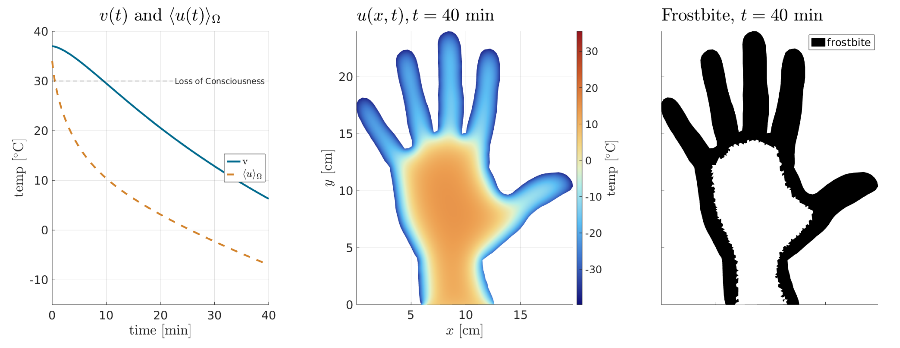
[1] *IT'IS Database for thermal and electromagnetic parameters of biological tissues*
[2] Taylor, et al. (2014); *Hands and feet: physiological insulators, radiators and evaporators.*
[3] Pennes (1948); *Analysis of Tissue and Arterial Blood Temperatures*
[4] Boron (2012); *Medical Physiology, 2nd Edition*
[5] Collins (1983); *Hypothermia: The Facts*

Linear Model: Simulation

Constant Coefficient (Linear) Model

$$R(u, v) = A(u - v)$$
$$S(u, v) = B(v - \langle u \rangle_\Omega)$$

$$\alpha(u, v) = \begin{cases} C_{\text{air}}(u - u_{\text{air}}), & u \in \partial\Omega_{\text{air}}, \\ C_{\text{wrist}}(u - v), & u \in \partial\Omega_{\text{wrist}}. \end{cases}$$



Theoretical Results: Weak and Discrete Schemes

Weak Form

We seek $u \in L^2(0,T;H^1(\Omega))$ with $\partial_t u \in L^2(0,T;H^1(\Omega)')$ and $v:(0,T] \to \mathbb{R}$ such that

$$\begin{aligned}
(c\,\partial_t u,\varphi) + (k\,\nabla u,\nabla\varphi) + (r(u,v),\varphi) - (\alpha(u,v),\varphi)_{\partial\Omega} &= (f,\varphi),\ \varphi \in H^1(\Omega), 0 < t \le T,\\
u(\cdot,0) &= u_{\text{init}} \in L^2(\Omega),\\
\kappa\frac{dv}{dt} + s(t,v,\langle u\rangle_\Omega) &= g, \quad 0 < t \le T,\\
v(0) &= v_{\text{init}}.
\end{aligned}$$

Fully Discrete Scheme

We discretize by order 1 Lagrange finite elements in space and by backward-Euler in time. That is, for each $t^n = n\tau$ for $tau > 0$, we seek $(U^n, V^n) \in V_h \times \mathbb{R}$ such that

$$\begin{aligned}
(c\,d_t U^n,\chi) + (k\,\nabla U^n,\nabla\chi) + (r(U^n,V^n),\chi) - (\alpha(U^n,V^n),\chi)_{\partial\Omega} &= (f^n,\chi), \quad \chi \in V_h,\\
d_t V^n + s(V^n,\langle U^n\rangle_\Omega) &= g^n,
\end{aligned}$$

$f^n := f(\cdot,t^n)$ and $g^n := g(t^n)$, $V_h \subset H^1(\Omega)$ is the test space, and d_t is the standard backward difference operator.

Theoretical Results: Stability Results

Theorem (Stability estimate) *Let Assumptions (a) through (d) hold and let the trace theorem hold with constant* C_{trace}*. Let also* (u, v) *be a (weak) solution of (1)-(2). Define*

$$\mathbf{E}(t) := (c\,u, u)_{L^2(\Omega)} + \kappa\, v^2, \qquad 0 < t \leq T.$$

Then there exist constants $A > 0$ *and* $C \geq 0$ *depending only on* k_{min}*,* L*,* C_{trace}*,* $|\Omega|$*, and* $|\partial\Omega|$ *such that*

$$\mathbf{E}(t) \leq e^{At}\left[\mathbf{E}(0) + C\,t + \int_0^t \left(\|f(s)\|^2 + |g(s)|^2\right)\, ds\right].$$

Corollary (Bound on $\frac{dv}{dt}$)
Under the assumptions of the previous theorem, there exist constants $C_1, C_2 > 0$*, depending only on* c_{min}*,* k_{min}*,* κ*,* L*,* C_{trace}*,* $|\Omega|$*,* $|\partial\Omega|$*,* T*, and data* $\|u_{init}\|_{L^2(\Omega)}$*,* $|v_{init}|$*,* $\|f\|_{L^2(0,T;L^2(\Omega))}$*,* $\|g\|_{L^\infty(0,T)}$*, such that*

$$\sup_{t\in[0,T]} \left|\frac{dv}{dt}\right| \leq C_1 + C_2 e^{AT/2}.$$

Theoretical Results: Discrete Results

Theorem (Discrete stability estimate) Let (U^n, V^n) be the solution of the discrete problem with initial data $(u_{\text{init},h}, v_{\text{init}})$. Set

$$\mathbf{E}^n := (c^n U^n, U^n)_{L^2(\Omega)} + \kappa (V^n)^2, \qquad 0 < n \le N_T.$$

Then, under the same prior assumptions, there exist constants $A > 0$ and $C > 0$ depending only on k_{min}, L, C_{trace}, $|\partial\Omega|$, and $|\Omega|$ such that for all $0 \le n \le N_T$:

$$\mathbf{E}^n \le \frac{\mathbf{E}^0}{(1-\tau A)^n} + \tau \sum_{k=1}^{n} \frac{\|f^k\| + |g^k| + C}{(1-\tau A)^{n-k+1}}, \quad \text{assuming } \tau < \frac{1}{A}. \tag{9}$$

Theorem (A–priori error estimate) *Under the same assumptions as previous theorems, for every $h > 0$ and τ sufficiently small the total error $\mathbf{e}^n$ satisfies*

$$\|\mathbf{e}\|_{L^\infty_\tau(0,T;\mathcal{X})} \le C(h^k + \tau), \tag{10}$$

where the constant $C > 0$ depends on c_{min}, c_{max}, κ, k_{min}, L, $\|u\|_{L^2(0,T;H^k(\Omega))}$, $\|\partial_t u\|_{L^2(0,T;H^k(\Omega))}$, $\|\partial_{tt} u\|_{L^2(0,T;L^2(\Omega))}$, $\|\frac{d^2 v}{dt^2}\|_{L^2(0,T)}$, the mesh–shape regularity, and on T, but not on h or τ.

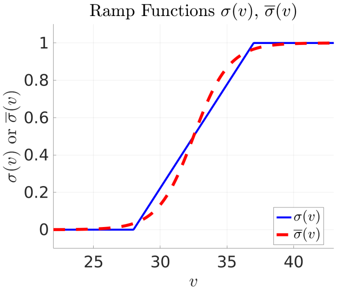
Nonlinear Model: Nonlinear Coefficients

Ramp Functions

Let $\sigma : \mathbb{R} \to [0,1]$ be continuous, monotonically increasing, and s.t.

$$\lim_{x \to -\infty} \sigma(x) = 0 \quad \text{and} \quad \lim_{x \to +\infty} \sigma(x) = 1. \tag{11}$$

Examples



Clipping Functions

For $M > 0$, let $T_M : \mathbb{R} \to [-M, M]$ be defined

$$T_M(x) = \max\{-M, \min\{x, M\}\}. \tag{12}$$

We write $\tilde{u} = T_M(u)$.

Nonlinear Coefficients

Let

$$r(u, v) = A\sigma(\tilde{u})\sigma(v), \tag{13a}$$

$$s(v, \langle u \rangle_\Omega) = B\sigma(v)\sigma(\langle u \rangle_\Omega), \tag{13b}$$

and let

$$R(u, v) = r(u, v)(\tilde{u} - v), \tag{14a}$$

$$S(v, \langle u \rangle_\Omega) = s(u, v)(v - \langle u \rangle_\Omega). \tag{14b}$$

The earlier stability results can be used to show that (14) are Lipschitz.

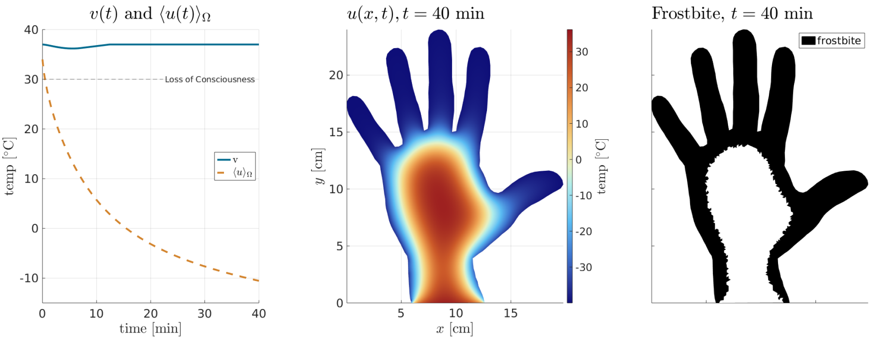
Nonlinear Model: Simulation

Nonlinear Coefficient Model

$$R(u,v) = r(u,v)\,(\tilde{u} - v), \qquad \text{where} \quad r(u,v) = A\,\sigma(\tilde{u})\,\sigma(v), \tag{15a}$$

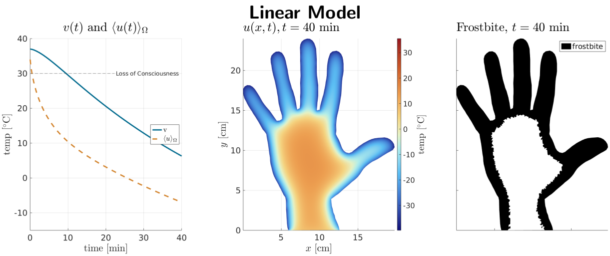
$$S(u,v) = s(u,v)\,(v - \langle u\rangle_\Omega), \qquad \text{where} \quad s(u,v) = B\,\sigma(\langle u\rangle_\Omega)\,\sigma(v). \tag{15b}$$

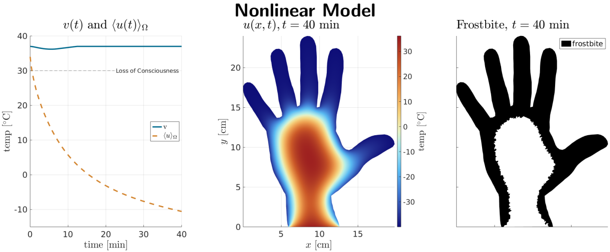
All parameters (including A and B) as in Table 1.



Note: $v(t) \leq v_{\text{init}}$ is enforced using a constraint operator.

Nonlinear Model: Comparison with Linear Model





Note: $v(t) \leq v_{\text{init}}$ is enforced using a constraint operator.

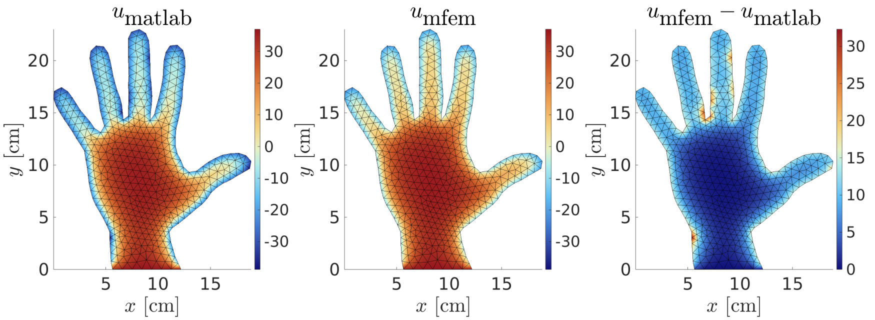
MFEM Results: Comparison of MFEM and MATLAB

Simplifications:

- $\nu = 37$ (i.e. system decoupled)
- Elliptic (not parabolic)
- Constant coefficient (linear) model

Features Retained:

- Mixed BC's (Robin and Dirichlet)
- Realistic domain geometry



Comparison of MFEM and MATLAB Results

Summary

Coupled PDE–ODE Model: Bioheat transfer in tissue (u) via PDE and core temperature (v) via ODE, coupled through R, S and boundary condition α.

Energy Stability: Established energy functional $E(t) = \frac{1}{2}\left(\|u\|_{L^2}^2 + |v|^2\right)$, yielding stability bound and $\sup |dv/dt| < \infty$.

A Priori Estimate: Fully discrete backward-Euler–Galerkin scheme satisfies

$$\max_n\{\|u(\cdot, t_n) - U^n\|_{L^2(\Omega)} + |v(t_n) - V^n|\} \leq C\,(h^2 + \Delta t).$$

Nonlinear Extension: Introduced ramp coefficients to define $r(u, v), s(v, \langle u \rangle_\Omega)$; shows physiologically reasonable behavior.

Simulation: Compared linear vs nonlinear dynamics; compared MFEM and MATLAB on a simplified problem.