Implementation of IMEX Integrators in MFEM with Examples

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Outline

Motivation

- **Context.** Implicit-Explicit (IMEX) schemes are used extensively to solve time dependent ODEs, especially in the context of fluid and heat flow.
- **Idea.** Consider a time-dependent ODE of the form:

$$\frac{du}{dt} = f_1(u,t) + f_2(u,t).$$

  Let f_1 be difficult to integrate implicitly (possibly non-linear). Let f_2 be stiff. An IMEX scheme applies an explicit integration scheme to f_1 and implicit to f_2.
- **Examples.**
  - **Forward-Backward Euler:**
    $u_{n+1} = u_n + \Delta t(f_1(u_n, t_n) + f_2(u_{n+1}, t_{n+1}))$
  - **CNAB-3:**
    $u_{n+1} = u_n + \frac{3\Delta t}{2} f_1(u_n) - \frac{\Delta t}{2} f_1(u_{n-1}) + \frac{\Delta t}{2} f_2(u_{n+1}) + \frac{\Delta t}{2} f_2(u_n)$

Motivation - Advection Diffusion Example

- **Advection-Diffusion.** Consider the 2-D Advection Diffusion problem:

$$\frac{du}{dt} = \nabla \cdot \kappa \nabla u - \beta \cdot \nabla u.$$

- In the context of a finite element spatial discretization, the advection term is not PSD. The diffusion term may be stiff, but is PSD. So one might use an implicit scheme on the diffusion term, and an explicit scheme on the advection term.
- **CFL Condition.** Using an IMEX Scheme, the CFL condition is $\Delta t \leq C\Delta x$ instead of $\Delta t \leq C(\Delta x)^2$ for an explicit scheme.

Implementation - SplitTimeDependentOperator

- **TimeDependentOperator** In MFEM, TimeDependentOperators are of the form

$$(u, t) \to k(u, t),$$

where k solves the equation:

$$F(u, k, t) = G(u, t).$$

In the case of differential equations, $k = \frac{du}{dt}$.

- **Split Operators** We implement a new class, SplitTimeDependentOperators, which are of the form

$$(u, t) \to k(u, t),$$

where k solves the equation:

$$F(u, k, t) = G_1(u, t) + G_2(u, t).$$

Implementation - SplitTimeDependentOperator Methods

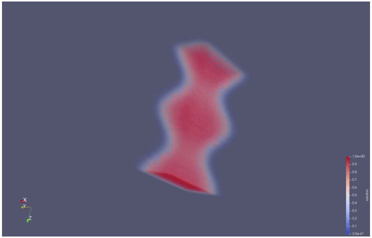
- The TimeDependentOperator class has important methods relevant to us:
  - **Mult** Performs the operation $(u, t) \to k(u, t)$
  - **ImplicitSolve** Given a scalar γ, solves $F(u + \gamma k, k, t) = G(u + \gamma k, t)$
- Meanwhile SplitTimeDependentOperator uses the following:
  - **Mult1** Performs the operation corresponding to G_1
  - **Mult2** Performs the operation corresponding to G_2
  - **ImplicitSolve1** Solves $F(u + \gamma k, k, t) = G_1(u + \gamma k, t)$
  - **ImplicitSolve2** Solves $F(u + \gamma k, k, t) = G_2(u + \gamma k, t)$

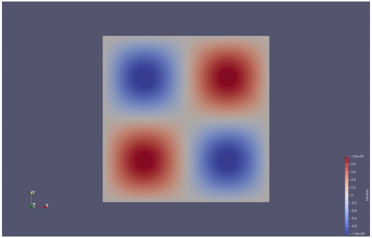
Implementation - SplitODESolver

- **Split Solvers** We implement a SplitODESolver class which inherits from ODESolver.
  - Solves systems of the form $\frac{du}{dt} = f_1(u,t) + f_2(u,t)$.
- **Example Solvers** We have currently implemented the following schemes:
  - **Forward Backward Euler**
  - **RK (2,2,2)** 2 implicit stages, 2 explicit stages, 2nd order.
  - **RK (2,3,2)** 2 implicit stages, 3 explicit stages, 2nd order.
  - **RK (3,4,3)** 3 implicit stages, 4 explicit stages, 3rd order.

Application - Time Dependent Advection-Diffusion MFEM Example Code

- **New Example** We write a new example code to demonstrate the use of the new IMEX Solvers. This code can solve a Time Dependent Advection-Diffusion problem on 2-D and 3-D meshes. The examples shown below use periodic boundary conditions.
- **Spatial Discretization** The spatial discretization is given by an interior penalty DG method.





Thank you for your attention. Any questions?

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