2025
MFEM
Community
Workshop

EMORY UNIVERSITY

A Domain-Decomposition Framework for Multiphysics Biomedical Modeling: Application to Cardiac Radiofrequency Ablation

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Outline

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01.

Clinical Background

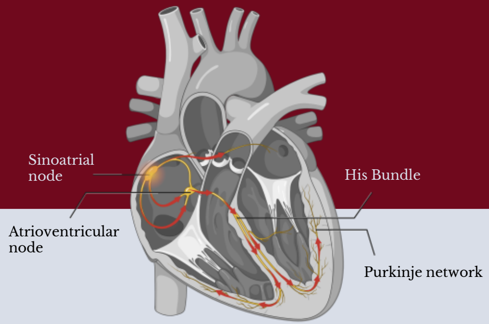
Cardiac **Arrhytmias**, Radiofrequency **Ablation** and Motivation to **Modeling**

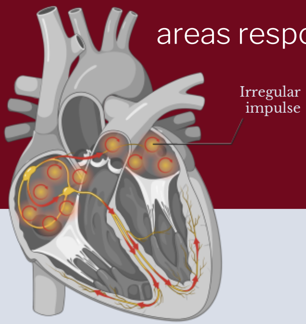
Cardiac RadioFrequency Ablation

Arrhytmia

“Any problem in the rate or rhythm of a person’s heartbeat”

- *American Heart Association*



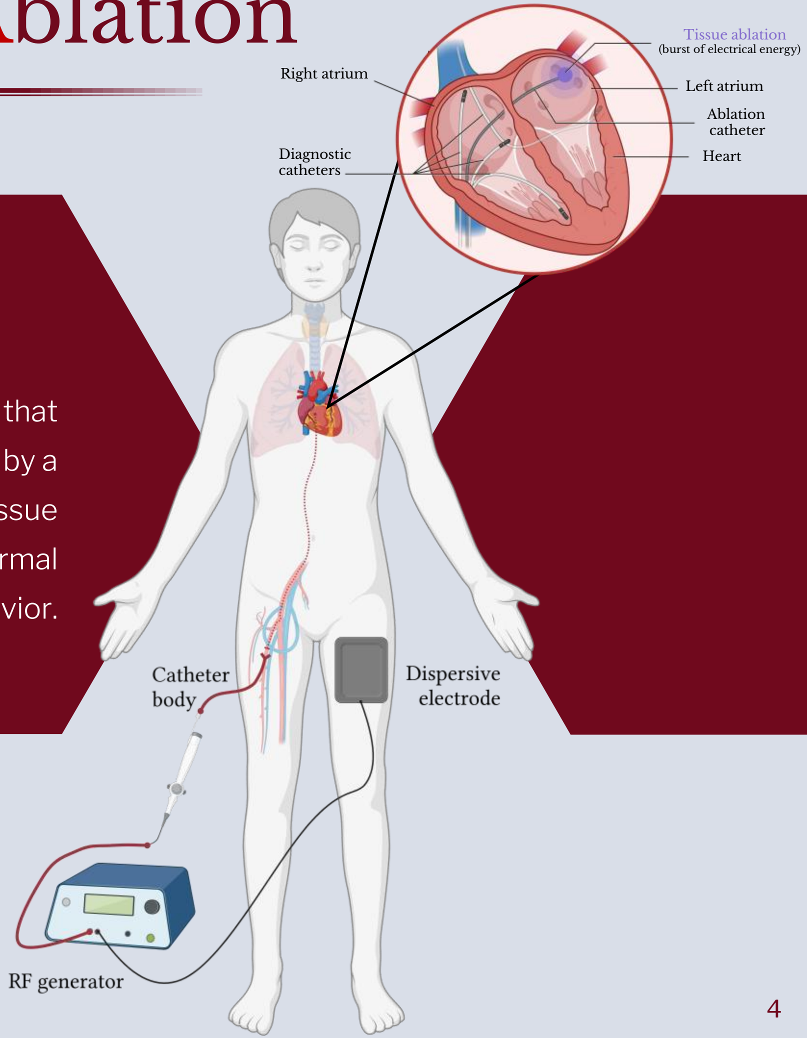


Typical Heart Rythm

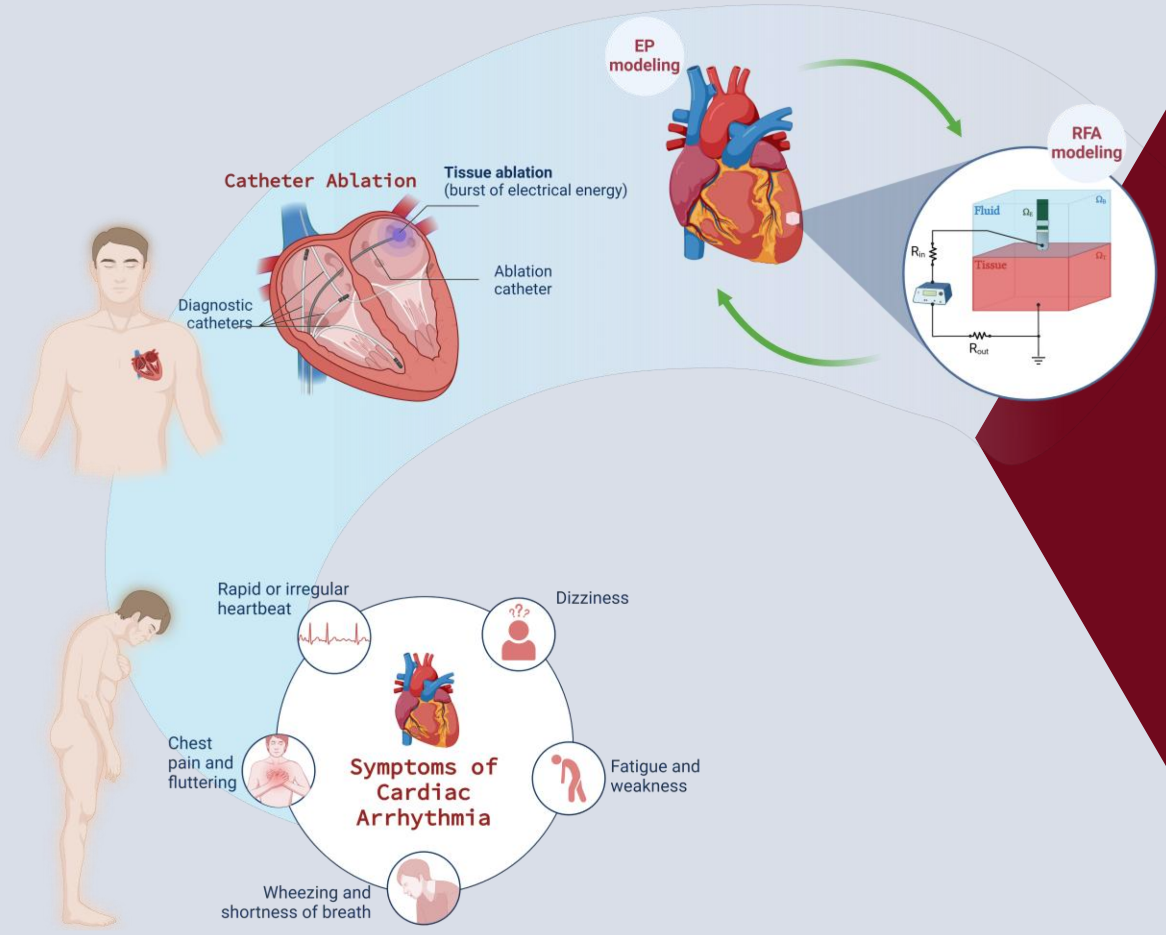
Atrial Fibrillation (AFib)

Cardiac Ablation

Minimally invasive treatment that delivers **energy** (heat, cold) by a catheter to destroy targeted tissue areas responsible for abnormal electrical behavior.



Why do we need modeling?



01. Improve the **mechanistic understanding** of lesion formation

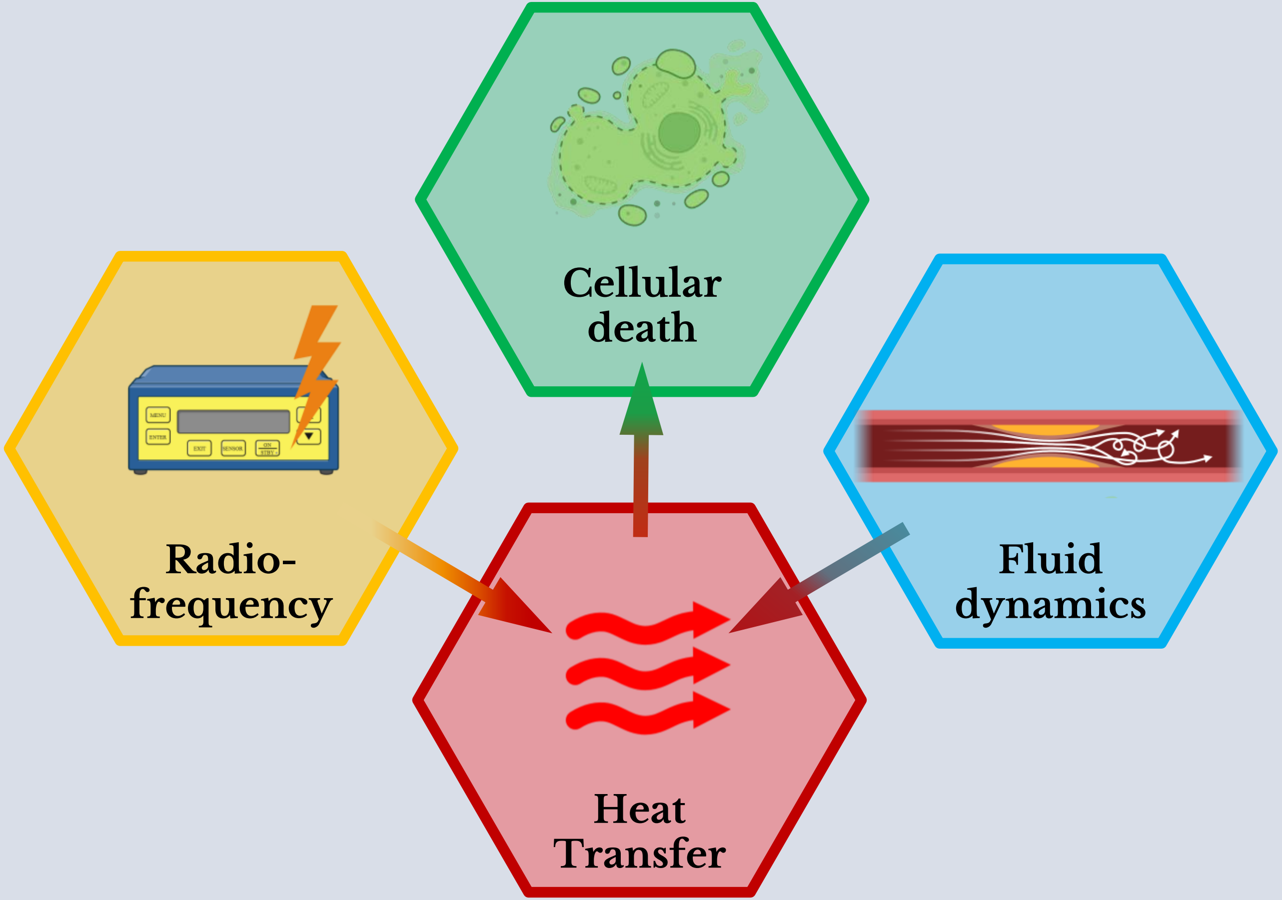
02. **Patient-tailored** therapy

03. **Optimal design** of new devices and clinical protocols

02. Multiphysics Model of RFA

Mathematical and **computational models** of *RF*, *Bioheat*, *Cellular death*, *Fluid Dynamics*

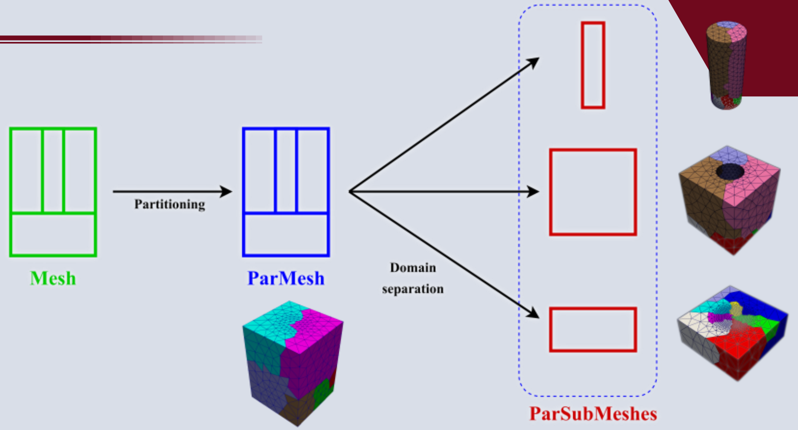
Model overview

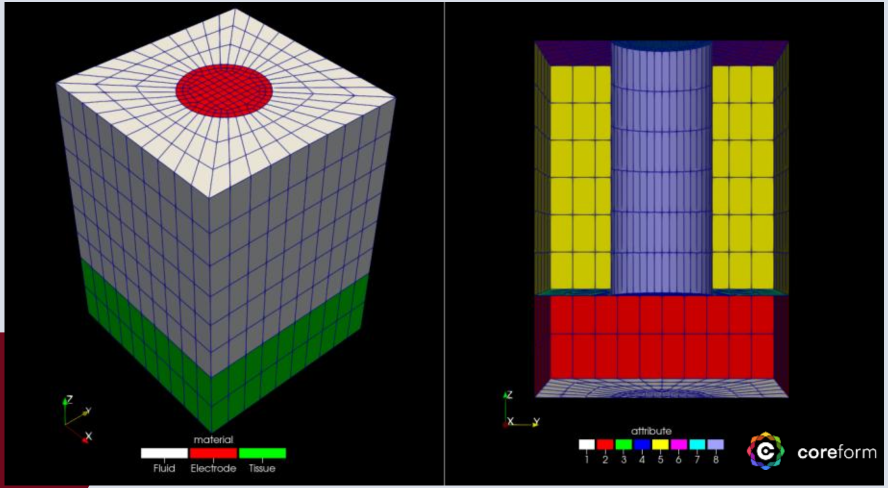


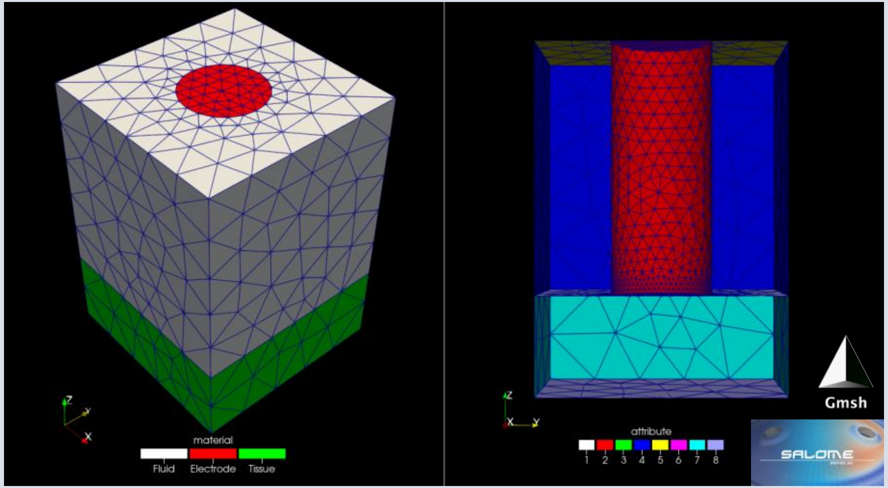
Geometry

Idealized geometry (...for now)

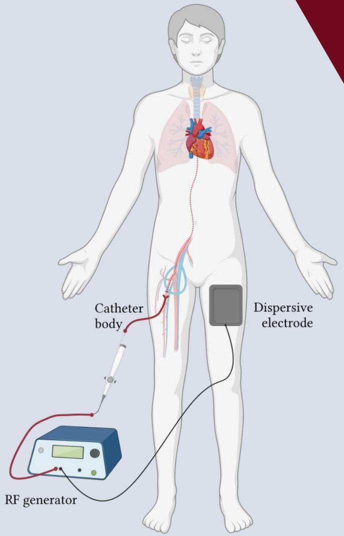
- Three physical domains: Tissue, Fluid, Electrode
- Available both as *structured* and *unstructured* meshes







Radiofrequency source

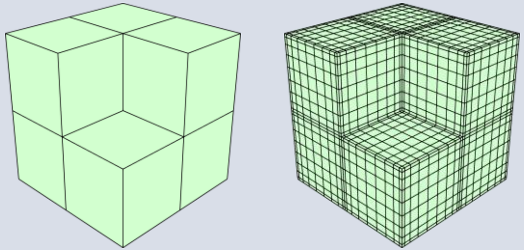


Quasi-static Maxwell's equation

$$\nabla \cdot (\boldsymbol{\sigma} \nabla \Phi) = 0 \qquad \text{in } \Omega_\mathsf{f}, \Omega_\mathsf{t}$$

Numerical implementation

- Spatial discretization: $\boldsymbol{Q_4}$ elements
- Linear solver: `CG` preconditioned with `Low-Order-Refined`[1] (**LOR**) coupled to AMG (PA)



High-order (FE) mesh

Low-order (FE) refined mesh

t = Tissue
f = Fluid
e - Electrode

[1] Will Pazner, Tzanio Kolev, and Jean-Sylvain Camier. End-to-end gpu acceleration of low-order-refined preconditioning for high-order finite element discretizations. The International Journal of High Performance Computing Applications, 37(5):578–599, 2023.
[2] Pazner, W. "Efficient low-order refined preconditioners for high-order matrix-free continuous and discontinuous Galerkin methods, arXiv preprint 1908.07071 (Aug. 2019)." *URL https://arxiv. org/abs* (1908).

Heat Transfer

Pennes' bioheat equation

$$\rho_t c_t \frac{\partial T}{\partial t} = \nabla \cdot (\mathbf{k}_t \nabla T) + Q_m + Q_p + Q_s \qquad \text{in } \Omega_t$$

$$\rho_f c_f \frac{\partial T}{\partial t} = \nabla \cdot (\mathbf{k}_f \nabla T) - \mathbf{u} \cdot \nabla T \qquad \text{in } \Omega_f$$

$$\rho_e c_e \frac{\partial T}{\partial t} = \nabla \cdot (\mathbf{k}_e \nabla T) \qquad \text{in } \Omega_e$$

$$Q_s = \boldsymbol{\sigma} \cdot |\mathbf{E}|^2 \quad [\mathrm{W/m^3}]$$

Q_m = metabolic (constant)

$Q_p = -\mathrm{w_b} \rho_b c_b (T_t - T_{bp})$

T_{bp} constant temperature of blood perfusing the tissue

w_b is a perfusion coefficient depending on the cell viability so that $0 < \mathrm{w_b(N)} \leq \mathrm{w_{b,max}}$

Numerical implementation

- Spatial discretization: $\boldsymbol{Q_2}$ elements
- Time discretization: two-step third-order Singly-diagonal implicit Runge-Kutta scheme (*SDIRK23*)
- Linear solver: `GMRES` preconditioned with matrix-free `Jacobi smoother` (PA)

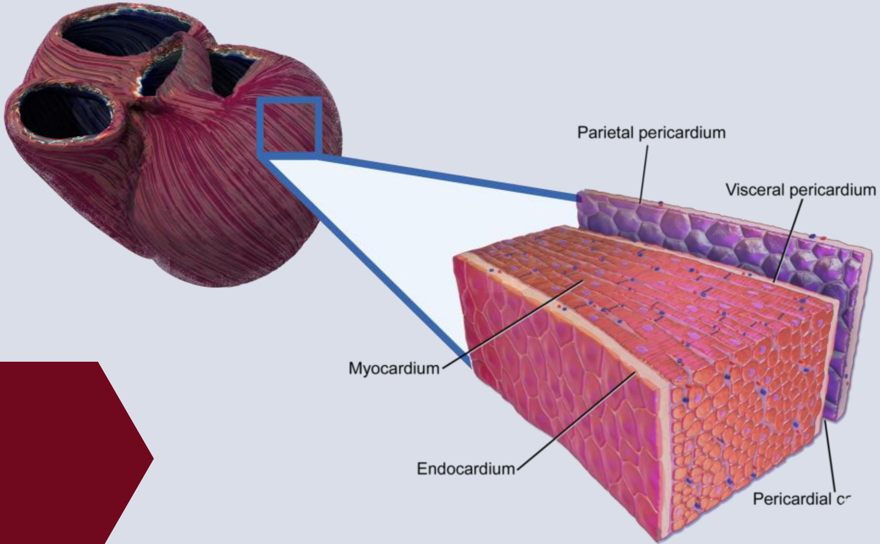
t = Tissue
f = Fluid
e - Electrode

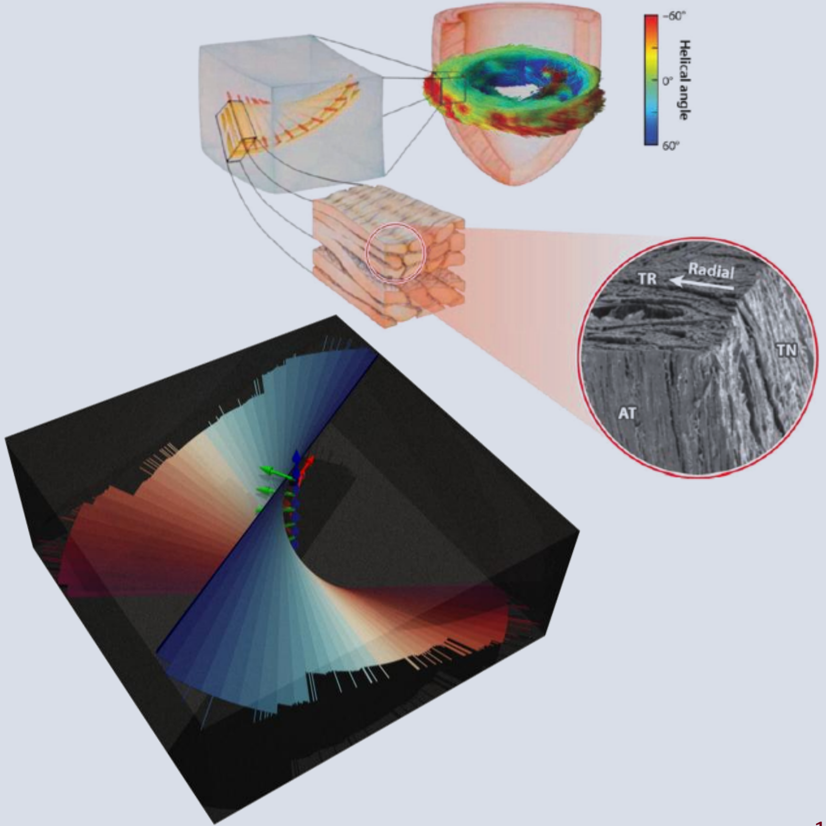
Anisotropy

- **Anisotropic thermo-electrical** conductive properties in the tissue due to **fiber microstructure**

$$\boldsymbol{\sigma} = \mathbf{R} \begin{bmatrix} \sigma_f & 0 & 0 \\ 0 & \sigma_\perp^y & 0 \\ 0 & 0 & \sigma_\perp^z \end{bmatrix} \mathbf{R}^T \qquad \mathbf{k} = \mathbf{R} \begin{bmatrix} k_f & 0 & 0 \\ 0 & k_\perp^y & 0 \\ 0 & 0 & k_\perp^z \end{bmatrix} \mathbf{R}^T$$

$$R = R_z(\alpha)\, R_y(\beta)\, R_x(\gamma)$$





Cellular death

Vulnerable (Unfolded)

Alive (Native) $\quad N \underset{\upsilon_3}{\overset{\upsilon_1}{\rightleftharpoons}} U \xrightarrow{\upsilon_2} D \quad$ Dead (Denatured)

Three-state cell-death model

$$\frac{dN}{dt} = -\upsilon_1 N + \upsilon_3 U$$
$$\frac{dU}{dt} = \upsilon_1 N - \upsilon_2 U - \upsilon_3 U \qquad \text{in } \Omega_t$$
$$\frac{dD}{dt} = \upsilon_2 U$$
$$\upsilon_i(T) = A_i e^{-\Delta E/RT}$$
$$N + U + D = 1$$

Numerical implementation

- **Eigenvalue method** to solve $\frac{dX}{dt} = \mathcal{A}\,\mathbf{X}, \quad \mathbf{X}^T = [N, U, D]^T$

$$X = \mathbf{P}\, e^{\int_0^t diag(\lambda) d\tau} \mathbf{P}^{-1} \mathbf{x}(0)$$

$$\mathcal{A} = \begin{bmatrix} -\upsilon_1 & \upsilon_3 & 0 \\ \upsilon_1 & -(\upsilon_2 + \upsilon_3) & 0 \\ 0 & \upsilon_2 & 0 \end{bmatrix} = \mathbf{P}\,\Lambda\,\mathbf{P}^{-1} \qquad \mathbf{X}_{n+1} = \mathbf{P}\, e^{\Lambda\,\Delta t} \mathbf{P}^{-1} X_n$$

- **Symbolic evaluation** of eigenpairs $\{\Lambda, \mathbf{P}\}$

t = Tissue

Fluid Dynamics

Incompressible Navier Stokes equation

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}^* \cdot \nabla \mathbf{u} - \nabla \cdot \boldsymbol{\tau} + \nabla p = \mathbf{f}$$

$$\nabla \cdot \mathbf{u} = 0$$

$\boldsymbol{u}^* = \sum_i^p (\boldsymbol{\beta}_i\, \boldsymbol{u}_{n+1-i} \cdot \boldsymbol{\nabla})\boldsymbol{u}_{n+1}$ **extrapolated velocity** (semi-implicit convection)

$\boldsymbol{\tau}$ **viscous stress tensor** $\boldsymbol{\tau} = \begin{cases} \boldsymbol{\nu}\, \boldsymbol{\nabla}\mathbf{u} & \text{Stiff strain} \\ \boldsymbol{\nu}(\boldsymbol{\nabla}\mathbf{u} + \boldsymbol{\nabla}\boldsymbol{u}^T) & \text{Viscous strain} \end{cases}$

Numerical implementation

ALgebraic splitting Time ADaptive solver for Incompressible Navier-Stokes (ALADINS) [1]

- Time discretization: **BDF** up to order 3
- Spatial discretization: stable pair $\boldsymbol{Q_2} - \boldsymbol{Q_1}$
- Time adaptivity (Pressure correction- based)
- Algebraic system: $\mathcal{A} = \begin{bmatrix} \mathcal{C} & \mathcal{G} \\ \mathcal{D} & \end{bmatrix}$, with $\mathcal{C} = \frac{\alpha}{\Delta t}\mathcal{M} + \nu\mathcal{K} + \mathcal{N}(\boldsymbol{u})$, $\mathcal{G} = -\mathcal{D}^T$
- `FGMRES` preconditioned with **block preconditioner** $\widehat{\mathcal{P}}$

[1] Alessandro Veneziani and Umberto Villa. Aladins: An algebraic splitting time adaptive solver for the incompressible navier–stokes equations. Journal of Computational Physics, 238:359–375, 2013.

Fluid Dynamic: Preconditioner

Navier-Stokes **Block Preconditioner** $\hat{\mathcal{P}}$

- **Algebraic splitting pressure corrected** preconditioner[1,2]

$$\mathcal{A} \approx \begin{bmatrix} \hat{\mathcal{C}} & \\ \mathcal{D} & -\hat{\mathcal{S}} \end{bmatrix} \begin{bmatrix} \boldsymbol{I} & \mathcal{H}_2 \mathcal{G} \mathcal{R} \\ & \mathcal{Q} \end{bmatrix}$$

Where we assumed :

→ $\hat{\mathcal{S}}$ **preconditioner for the Schur Complement** $\boldsymbol{\Sigma} = \mathcal{D}\mathcal{C}^{-1}\mathcal{G}$, $\hat{\mathcal{S}} = \mathcal{D}\, diag(\mathcal{M}_v)^{-1}\mathcal{G}$

→ $\hat{\mathcal{C}}$ approximation to $\mathcal{C}$,

→ $\mathcal{H}_1, \mathcal{H}_2$ cheap approximation to $\mathcal{C}^{-1}$

→ $\mathcal{Q}, \mathcal{R}$ pressure correction matrices, depend on the chosen **factorization method.**

- **Factorization method:**
  High-Order Yosida (HOY)
  (momentum preserving)

$$\mathcal{A}_{HOY} = \begin{bmatrix} \hat{\mathcal{C}} & \\ \mathcal{D} & -\hat{\mathcal{S}} \end{bmatrix} \begin{bmatrix} \boldsymbol{I} & \mathcal{C}^{-1}\mathcal{G} \\ & \mathcal{Q} \end{bmatrix}$$

$$\mathcal{H}_1 = \left(\frac{\alpha}{\Delta t}\mathcal{M}\right)^{-1}, \quad \mathcal{H}_2 = \mathcal{C}^{-1}$$

$$\mathcal{R} = \boldsymbol{I}, \quad \mathcal{Q} = (\mathcal{D}\mathcal{H}_1\mathcal{C}\,\mathcal{H}_1\mathcal{G})^{-1}\hat{\mathcal{S}}$$

$$\boldsymbol{D}_k = -\mathcal{D}(-\mathcal{H}_1\mathcal{F})^k\mathcal{G}, \quad \mathrm{k} > 0 \qquad \mathcal{F} = \nu\mathcal{K} + \mathcal{N}(\boldsymbol{u})$$

Pressure correction step:
HOY

$$\boldsymbol{y} = \mathcal{Q}^{-1}\boldsymbol{x} = \begin{cases} \hat{\mathcal{S}}\boldsymbol{z}_1 = \boldsymbol{D}_1\boldsymbol{z}_0 \\ \hat{\mathcal{S}}\boldsymbol{z}_2 = \boldsymbol{D}_2\boldsymbol{z}_1 + \boldsymbol{D}_1\boldsymbol{z}_0 \\ \vdots \\ \boldsymbol{y} = \sum_{i=0}^{p} \boldsymbol{z}_i \end{cases}$$

- **Time adaptivity:**

Adaptive selection of time step based on *a posteriori* **pressure-correction based** error estimator

$$\Delta t = \chi\, \Delta t_{old}$$

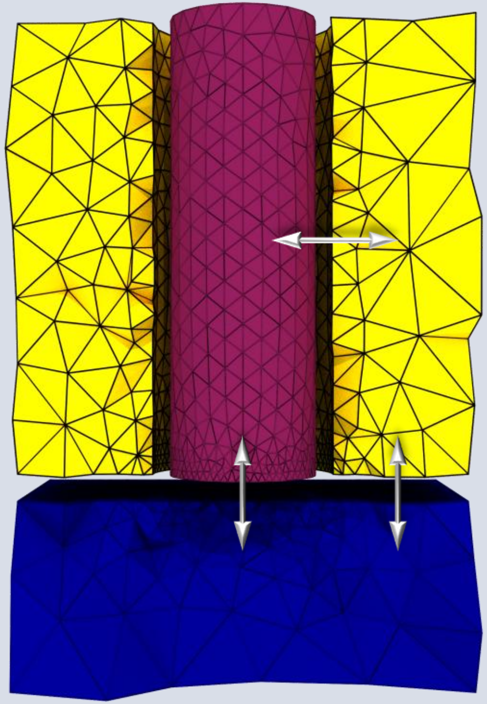
→ α **Safety factor** for time adaptivity

→ ϵ: **Tolerance** for time adaptivity

→ z_q pressure correction of order q

$$\chi = \min\left(\max\left(\alpha \cdot \left(\frac{\epsilon\, \Delta t_{old}}{\|\boldsymbol{z}_q\|}\right)^{\frac{1}{q}}, \chi_{min}\right), \chi_{max}\right)$$

[1] Alessandro Veneziani. Block factorized preconditioners for high-order accurate in time approximation of the navier-stokes equations. Numerical Methods for Partial Differential Equations, 19(4):487–510, 2003.
[2] Alessandro Veneziani and Umberto Villa. Aladins: An algebraic splitting time adaptive solver for the incompressible navier–stokes equations. Journal of Computational Physics, 238:359–375, 2013.

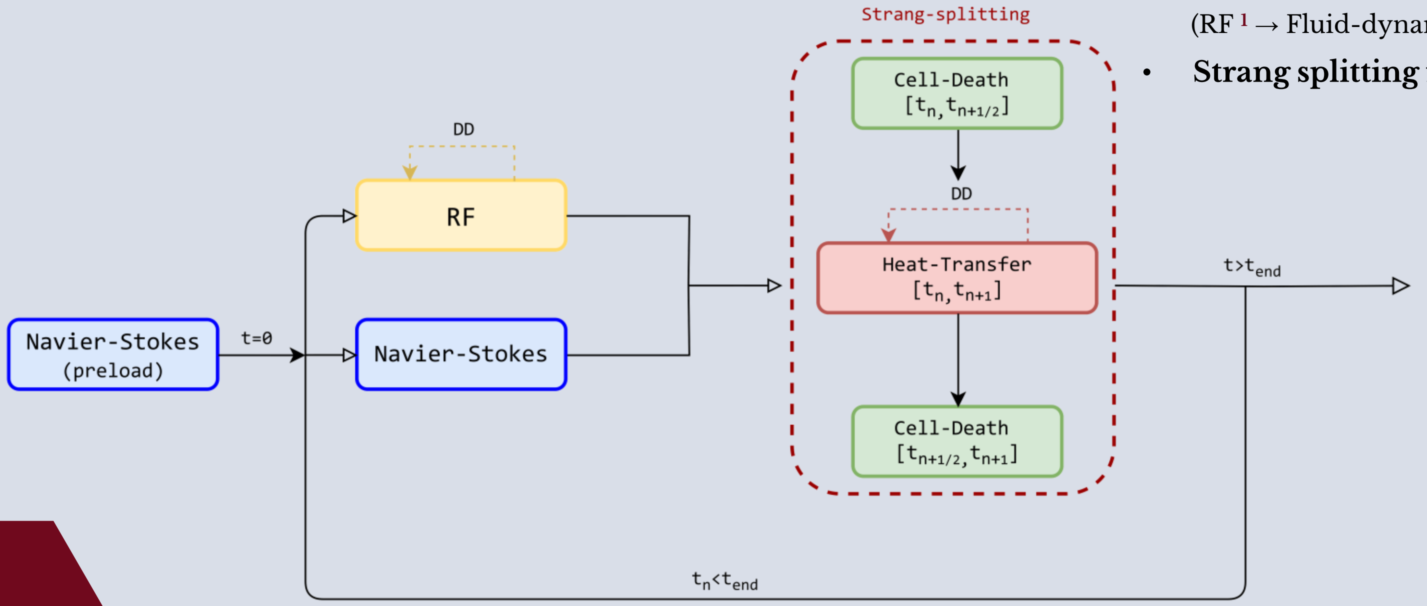


03.

Domain Decomposition

Modeling **interface** between *tissue*, *electrode* and *fluid* domains

Two-level segregation approach



1. Multiphysics segregation

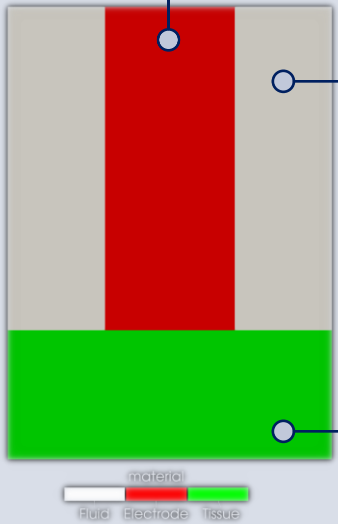
- **Staggered approach** for multiphysics (RF [1] → Fluid-dynamics →Heat transfer [1] →Cell-death)
- **Strang splitting** for HT/CD problems $O(\Delta t^2)$

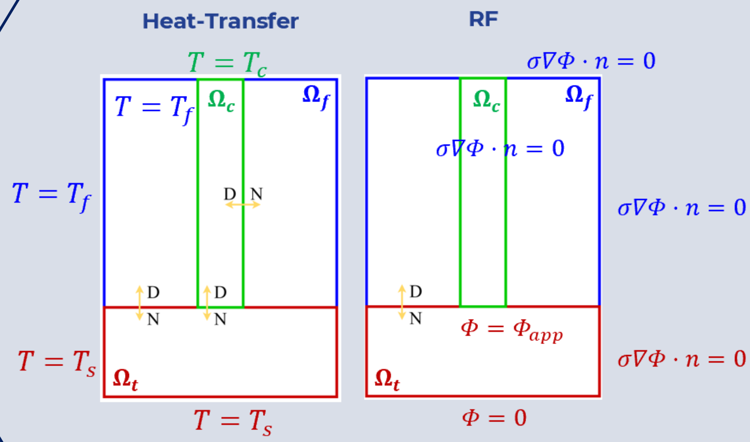
1. Solved until convergence of DD is achieved

Two-level segregation approach

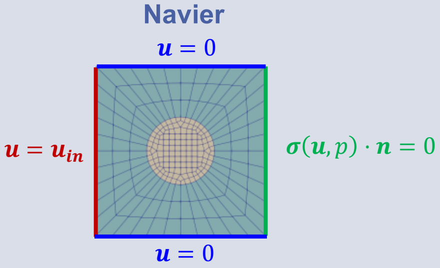
- Heat transfer

- RF
- Fluid dynamics
- Heat transfer (*advection-diffusion*)





- RF
- Heat transfer (*joule heating*)
- Cell-death



1. Multiphysics segregation

- **Staggered approach** for multiphysics

  (RF [1] → Fluid-dynamics →Heat transfer [1] →Cell-death)

2. Substructuring by (physical) subdomains

- **Nonoverlapping domain decomposition**

  Based on physical domains

- **Dirichlet-Neumann, Robin-Robin coupling (DN, RR)** [1]

  For transmission conditions at the physical interfaces

RF

$$\nabla \cdot \sigma_1 \nabla\phi_1^{n+1} = f_1 \quad \text{in } \Omega_1$$
$$\phi_1^{n+1} = \phi_2^{n} \quad \text{on } \Gamma_{12}$$

$$\nabla \cdot \sigma_2 \nabla\phi_2^{n+1} = f_2 \quad \text{in } \Omega_2$$
$$\sigma_2\nabla\phi_2^{n+1} \cdot \boldsymbol{n} = \sigma_1\nabla\phi_1^{n+1} \cdot \boldsymbol{n} \quad \text{on } \Gamma_{12}$$

1. Robin-Robin implemented and tested for RF subproblem

04.

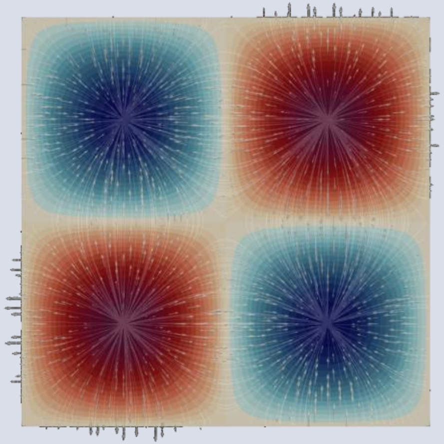
Results

Solvers verifications, Domain Decomposition convergence, RFA

Solvers verification: Heat transfer

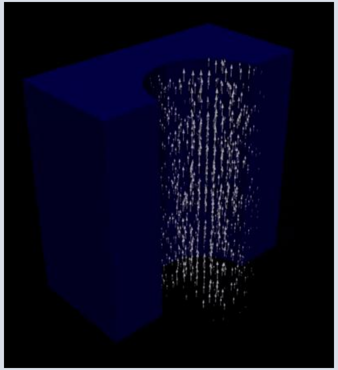
Convergence (MMS on squared domain) + toy problems

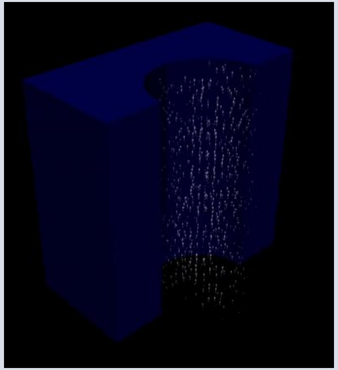
$$T(t) = \sin(k\,\pi\,x)\,\sin(k\,\pi\,y) + \beta t\,, \quad k = 2,\ \beta = 1.1$$



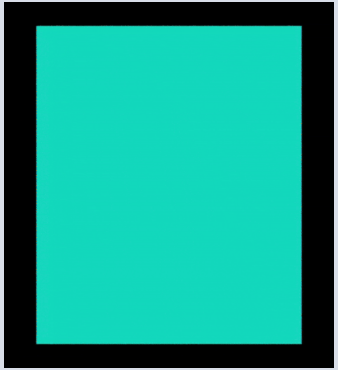
Heat-Transfer ($o = 2$)					
DOFs	h	L^2 error	L^2 rate	H^1 error	H^1 rate
81	0.25	0.01445	0	0.4044	0
289	0.125	0.001933	2.902	0.102	1.987
1089	0.0625	0.0002451	2.979	0.02553	1.998
4225	0.03125	3.075e-05	2.995	0.006383	2
16641	0.01562	3.847e-06	2.999	0.001596	2

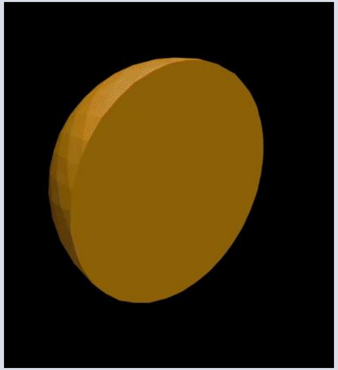
Advection-diffusion





Time-dependent square plate heating



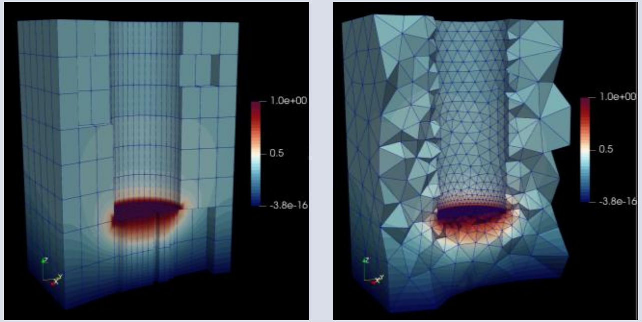


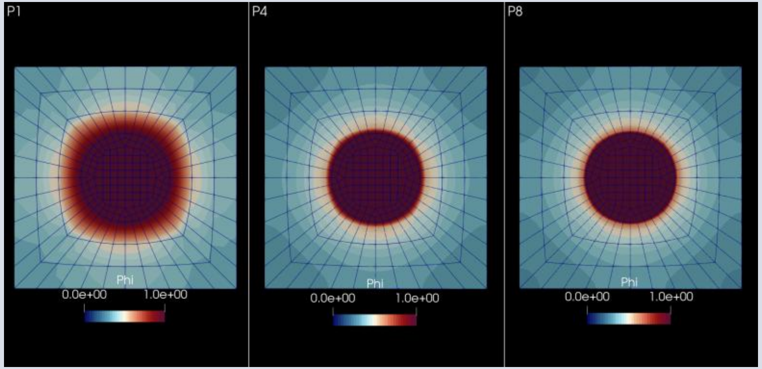
Convective cooling of sphere

Solvers verification: RF

- Convergence (MMS on squared domain) + toy problems

RF					
Dofs	h	L^2 error	L^2 rate	H^1 error	H^1 rate
289	0.2686	2.425×10^{-5}	0	0.001126	0
1089	0.1343	7.761×10^{-7}	4.965	7.144×10^{-5}	3.979
4225	0.06716	2.443×10^{-8}	4.99	4.478×10^{-6}	3.996
16641	0.03358	8.309×10^{-10}	4.878	2.802×10^{-7}	3.998

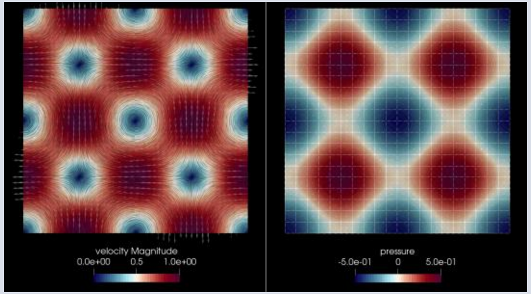


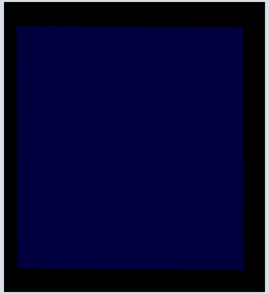


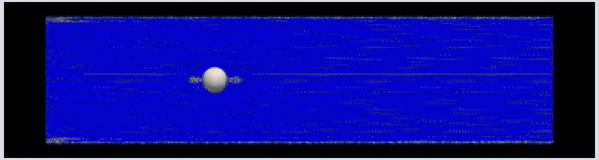
Solvers verification: Navier-Stokes

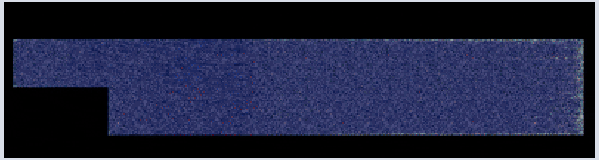
- Convergence (MMS on squared domain) + toy problems

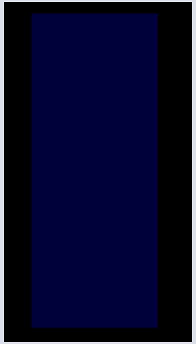
Navier-Stokes						
DOFs u	DOFs p	h	L^2 error (u)	L^2 error (p)	L^2 rate (u)	L^2 rate (p)
242	36	0.2	1.99644×10^{-2}	0.110601	0	0
882	121	0.1	5.26485×10^{-3}	0.0185581	1.92297	2.57525
3362	441	0.05	7.36945×10^{-4}	0.00379613	2.83676	2.28945
13122	1681	0.025	5.4351×10^{-5}	9.36223×10^{-4}	3.76118	2.0196
51842	6561	0.0125	4.3819×10^{-6}	2.99535×10^{-4}	3.63268	1.64413







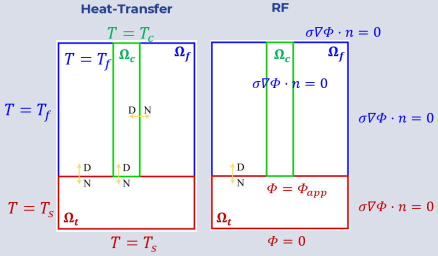


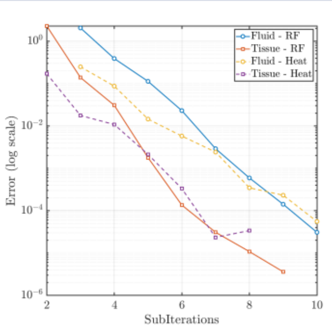


Domain Decomposition convergence

Subiteration convergence

Dirichlet-Neumann scheme exhibits **linear convergence** for Heat/RF problems.





h-refinement

Consistent results for DD convergence across different levels of **mesh refinement**
For RF, RR conditions did not lead to significant improvements (but we did not analyze the optimal parameters derived form Fourier Analysis)

h-Refinement				
Ref	Dofs RF	Dofs H	It-RF	It-Heat
0	$90k$	$19k$	10	7.4 ± 0.70
1	$673k$	$150k$	9	7.4 ± 0.52
2	$5M$	$1.1M$	9	7.7 ± 0.67

(a) Iteration count for DD convergence ($tol = 10^{-6}$) after **h-refinement** (uniform refinement). For heat problem, iterations taken over the first 10 timesteps. H = Heat, RF = RadioFrequency.

p-refinement

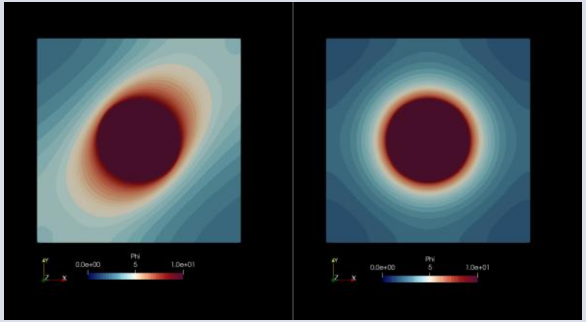
Slower convergence rates for the DD algorithm

p-Refinement				
Order (H-RF)	Dofs RF	Dofs H	It-RF	It-Heat
$Q_1 - Q_2$	$13k$	$3k$	10	7.5 ± 0.53
$Q_2 - Q_4$	$90k$	$15k$	10	7.4 ± 0.70
$Q_4 - Q_8$	$673k$	$150k$	15	12.9 ± 2.92

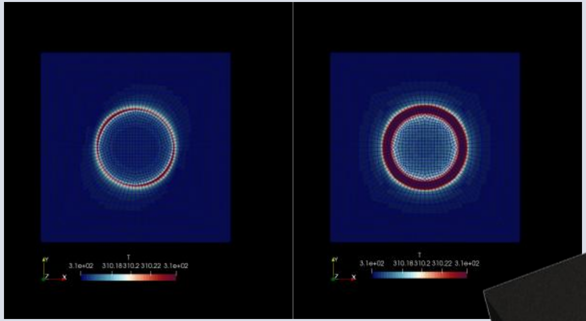
(b) Iteration count for DD convergence ($tol = 10^{-6}$) after **p-refinement**. For heat problem, iterations taken over the first 10 timesteps. H = Heat, RF = RadioFrequency.

Anisotropy

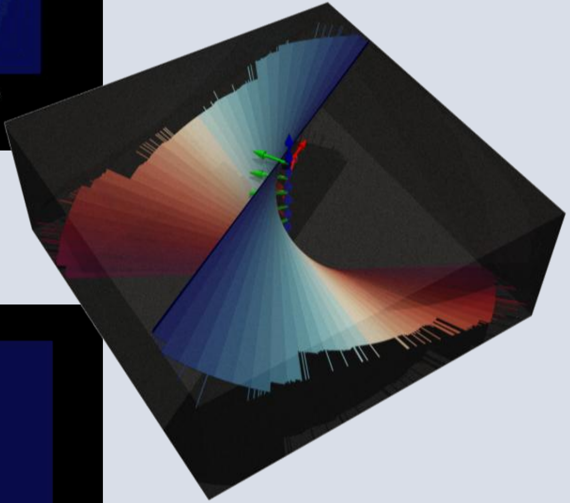
RF, tissue only



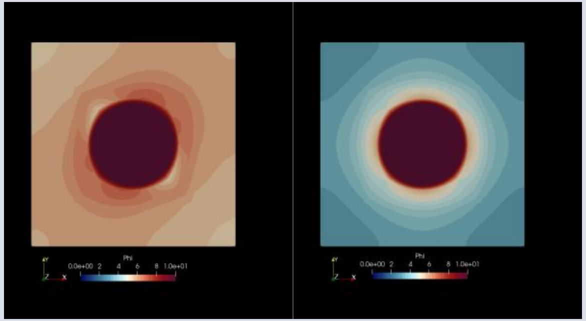
Electric potential



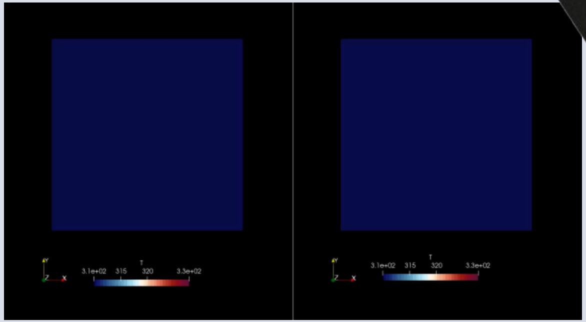
Electric field



RFA, three domains

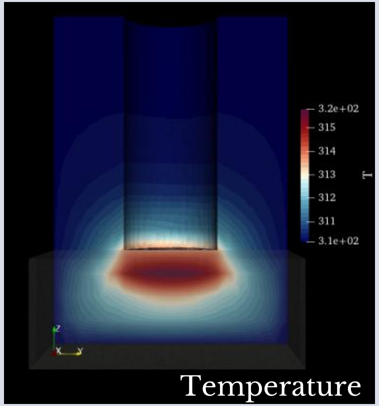


Electric potential

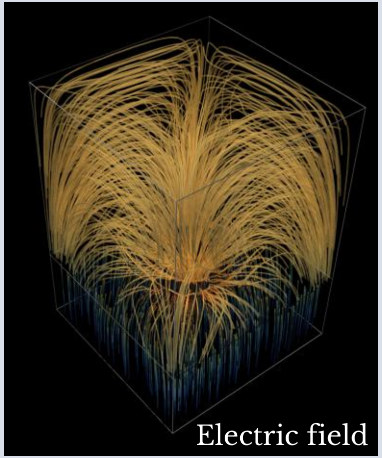


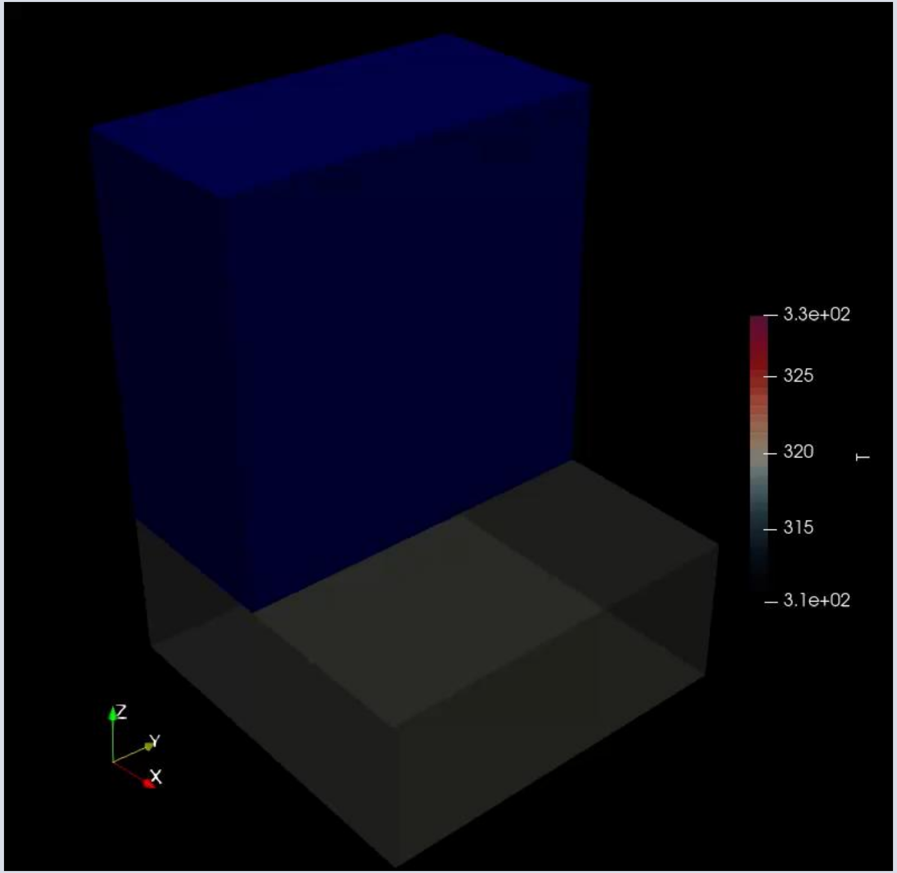
Temperature

(pseudo) RFA simulation



Temperature

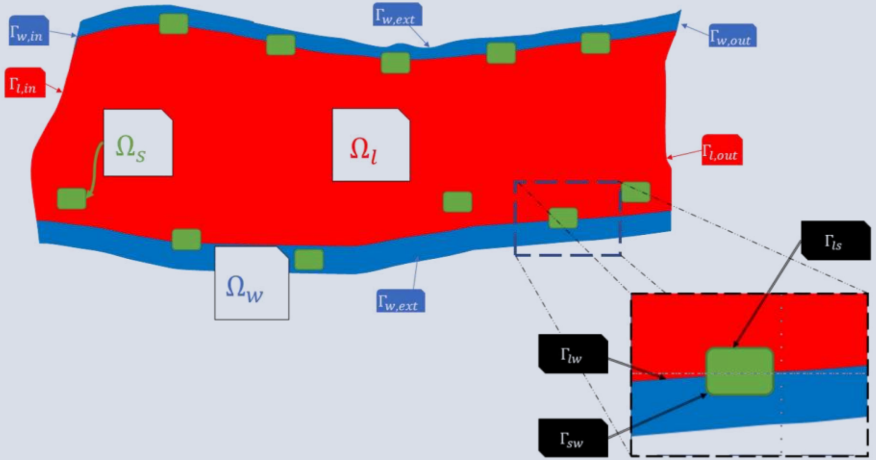

Electric field



05. Future work

Other applications

- **Different ablation sources:** MWA, HIFU, Laser
- **Different ablated tissues:** Liver, Intestine, Prostate, Bone
- **Erosion of bioresorbable stents**



1. Irving Martinez, Alessandro Veneziani. Multi-domain modeling and simulation of blood solutes and elution in stented arteries. Advances in Computational Science and Engineering, 2024, 2(2): 152-173. doi: 10.3934/acse.2024009

Future plan

Multiphysics model

- **1.1 Fluid dynamics:**
  - Strongly consistent stabilization (SUPG, GLS, VMS), and turbulence model (EFR)

$$\sum_{k\in\mathcal{T}} (\mathcal{R}_\phi, \tau \mathcal{L}_{stab}(w)),$$

$$\mathcal{L}_{stab} = \begin{cases} a\cdot\nabla & SUPG \\ -a\cdot\nabla + \nu\Delta & GLS \\ -a\cdot\nabla - \nu\Delta & VMS \end{cases}$$

- **1.2 Heat transfer:**
  - Extend tissue bioheat equation to **Non-Fourier models** (*SPL, DPL*) $\rho c \frac{\partial^2 T}{\partial t^2}$, $\rho c \nabla \cdot (k \frac{\partial \nabla T}{\partial t})$
- **1.3 Mechanics:**
  - Develop quasi-incompressible, hyperelastic, anisotropic **model for tissue mechanics**
  - Develop **electrode-tissue contact model** (leverage **MFEM-Tribol** interface)

Improved computational efficiency

- **High fidelity simulations:** Extend PA to all solvers and enable GPU acceleration
- **Clinically-suitable simulations:** investigate use of **Reduced Order Modeling (ROM)** techniques (**libROM**)

Ultimate goal: Surgical planning/therapy optimization

- Identification of optimality criteria
- **ROM** techniques to accelerate identification of optimal solution
- Ex-vivo and Pre-clinical Validation

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Thank you!