



BROWN

Proximal Galerkin for Contact Mechanics

Noe Reyes Rivas¹

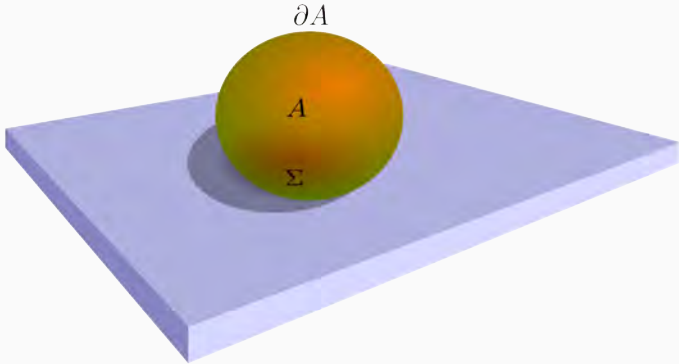
25 September 2026

Brown University

¹Advisor: Brendan Keith

Signorini's Problem

Signorini's Problem



What will be the equilibrium configuration of the orange elastic body resting on the blue rigid frictionless plane?

Signorini's Problem

We seek to minimize the strain energy functional

$$E : V \rightarrow \mathbb{R}, \quad E(u) = \frac{1}{2} \int_{\Omega} C \varepsilon(u) : \varepsilon(u) \, dx - \int_{\Omega} f \cdot u \, dx$$

over the feasible set

$$K = \{v \in V \mid \gamma v \cdot \tilde{n} \leq \phi \text{ a.e. on } \Gamma_C\},$$

where

$$V = \{v \in H^1(\Omega, \mathbb{R}^d) \mid \gamma v = g \text{ a.e. on } \Gamma_D\}.$$

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Remark: Throughout this talk, $C \varepsilon(u) = \lambda \operatorname{div}(u)I + 2\mu \varepsilon(u)$.

Numerical Methods Comparison

	feasible?	inf-dim?	mesh-indep?	no param to $0/\infty$?
active set/semismooth Newton	✓	✓	✗	✓
penalty/augmented Lagrangian	✗	✓	✓	✗
monotone multigrid	✓	✗	✓	✓
interior point	✓	✓	✓	✗

Credit: Patrick E. Farrell

Bregman Proximal Point

Definition (Legendre Function)

We say that a proper, convex, lower-semicontinuous function $R : \mathbb{R}^m \rightarrow \mathbb{R} \cup \{\infty\}$ is a *Legendre function* if

1. $\text{int}(\text{dom } R) \neq \emptyset$,
2. R is differentiable and strictly convex on $\text{int}(\text{dom } R)$, and
3. $|\nabla R(a^k)| \rightarrow \infty$ whenever $a^k \in \text{int}(\text{dom } R)$ and $a^k \rightarrow a \in \partial(\text{dom } R)$.

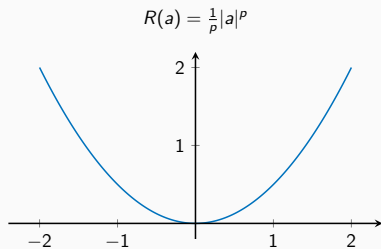
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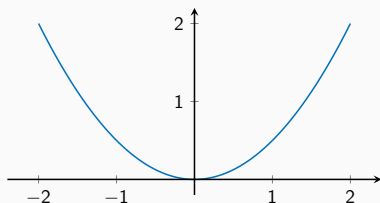
Remark: Property 3 is known as *essential smoothness*.

Examples of Legendre Functions

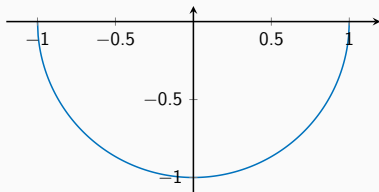


Examples of Legendre Functions

$$R(a) = \frac{1}{p}|a|^p$$

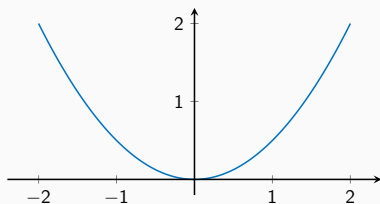


$$R(a) = -\sqrt{1 - |a|^2}$$

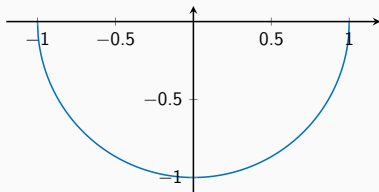


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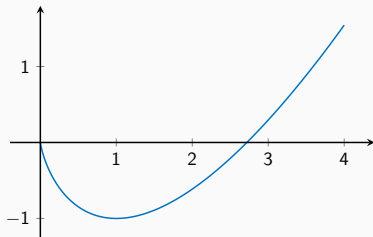
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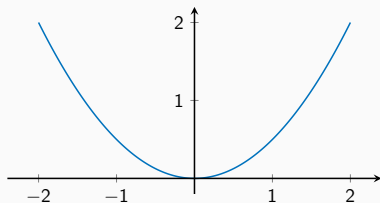


$$R(a) = a \ln a - a$$

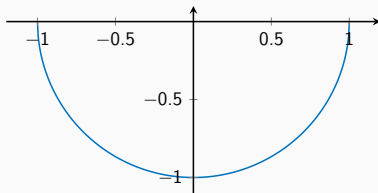


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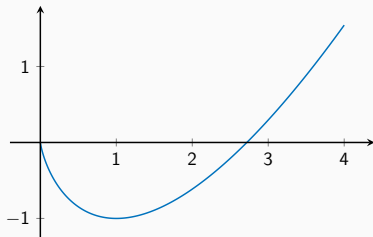
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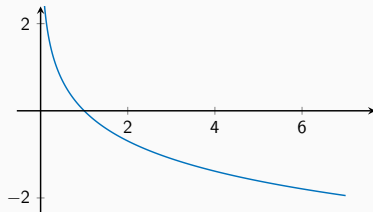
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Bregman Divergences

Definition (Bregman Divergence)

Let $R : \mathbb{R}^m \rightarrow \mathbb{R} \cup \{\infty\}$ be a Legendre function. The *Bregman divergence* associated to R is the function

$D_R : \text{dom } R \times \text{int}(\text{dom } R) \rightarrow [0, \infty)$ defined by

$$D_R(a, b) = R(a) - R(b) - \nabla R(b) \cdot (a - b).$$

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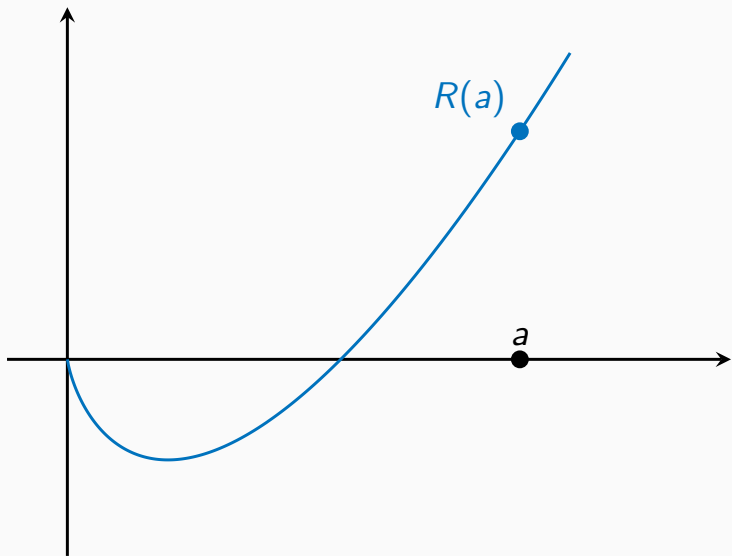
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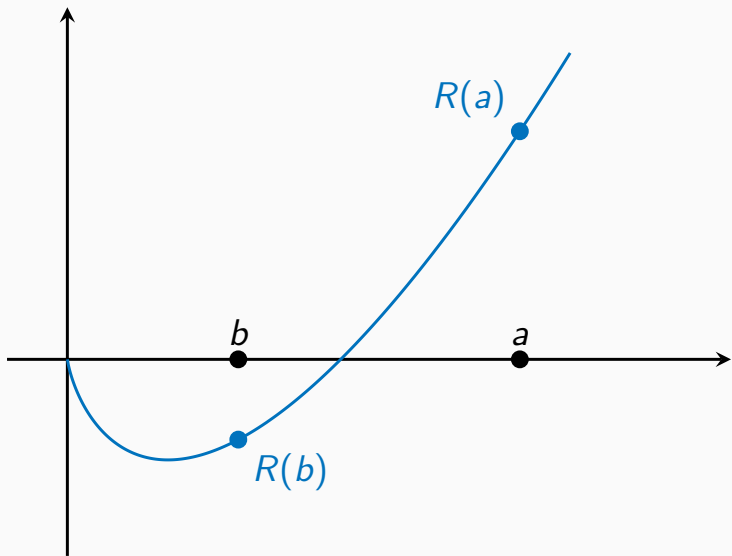
Bregman divergences satisfy the following properties:

1. If $a, b \in \text{int}(\text{dom } R)$, then $D_R(a, b) = 0$ if and only if $a = b$.
2. D_R is convex in its first argument (and strictly convex if restricted to $\text{int}(\text{dom } R) \times \text{int}(\text{dom } R)$).

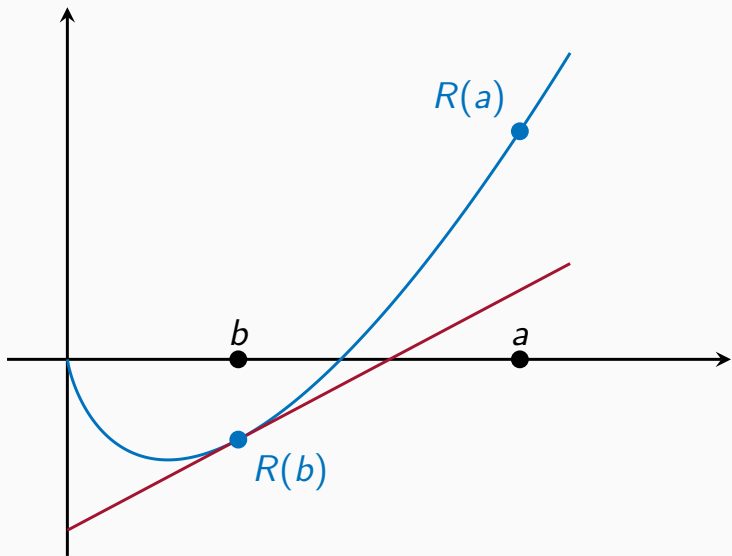
Bregman Divergences



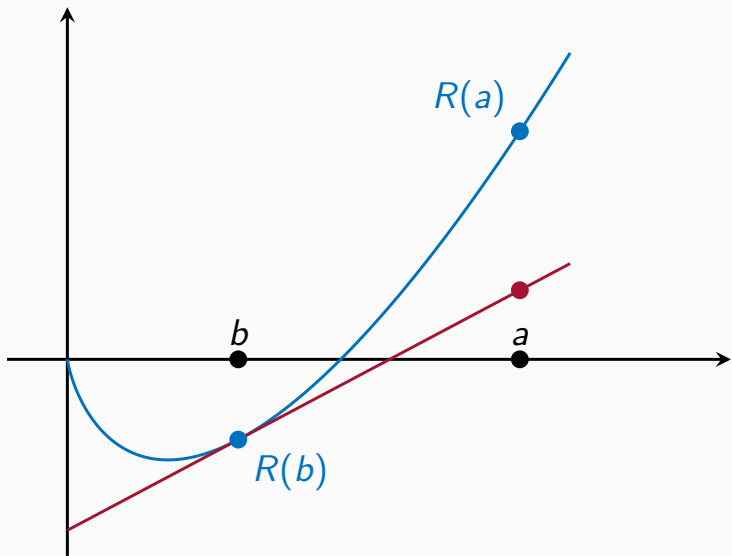
Bregman Divergences



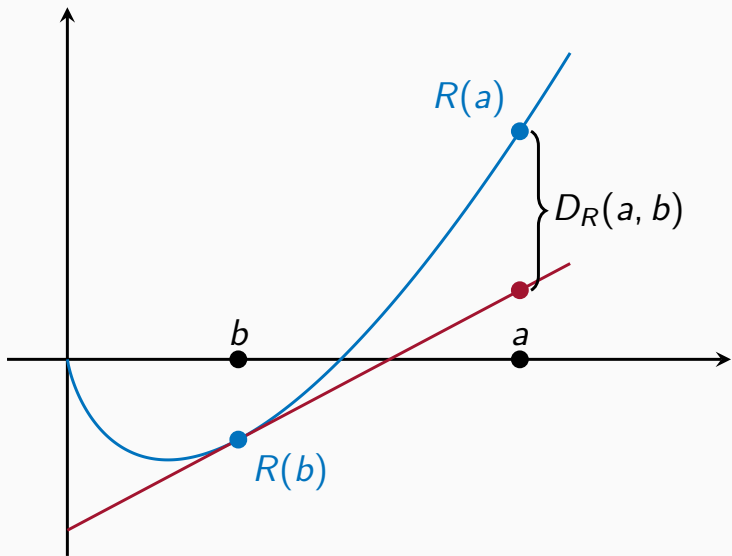
Bregman Divergences



Bregman Divergences



Bregman Divergences



The Bregman Proximal Point Method

Let $f : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{\infty\}$ be a proper, lower-semicontinuous, convex function and $\mathcal{C} \subseteq \mathbb{R}^n$ be non-empty, closed, and convex.

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Let $f : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{\infty\}$ be a proper, lower-semicontinuous, convex function and $\mathcal{C} \subseteq \mathbb{R}^n$ be non-empty, closed, and convex. To solve the optimization problem

$$\min_{x \in \mathcal{C}} f(x),$$

the Bregman proximal point method (BPP) generates a sequence $\{x^k\}_{k \in \mathbb{N}}$ such that

$$x^{k+1} = \arg \min_{x \in \mathcal{C}} \left\{ f(x) + \alpha_k^{-1} D_R(x, x^k) \right\},$$

where R is a Legendre function, D_R is the Bregman divergence, and $\alpha_k > 0$.

Convergence of Bregman Proximal Point

When does the sequence $\{x^k\}_{k \in \mathbb{N}}$ generated by BPP converge to a minimizer?

Theorem (Chen and Teboulle 1993)

Let x^ be a minimizer of f .*

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$$x^k \rightarrow x^* \quad \text{and} \quad f(x^k) \rightarrow f(x^*).$$

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In particular,

$$f(x^k) - f(x^*) = \mathcal{O}\left(1 / \sum_{j=0}^{k-1} \alpha_j\right).$$

Proximal Galerkin

We can apply Bregman proximal point to solve optimization problems over Sobolev spaces:

$$u^k = \arg \min_{u \in K} \left\{ E(u) + \alpha_k^{-1} \int_{\Omega_d} D_R(Bu, Bu^{k-1}) d\mathcal{H}_d \right\},$$

where...

- $B : V \rightarrow L^1(\Omega_d)$ is a bounded linear operator,
- $\Omega_d \subseteq \overline{\Omega}$ is measurable.

First order optimality yields

$$\text{Find } u^k \in V \text{ s.t. } \alpha_k E'(u^k) + B^* \nabla R(Bu^k) = B^* \nabla R(Bu^{k-1}) \quad \text{in } V'.$$

First order optimality yields

Find $u^k \in V$ s.t. $\alpha_k E'(u^k) + B^* \nabla R(Bu^k) = B^* \nabla R(Bu^{k-1})$ in V' .

Defining $\psi^k = \nabla R(Bu^k)$, we can rewrite this as a system: find $(u^k, \psi^k) \in V \times L^\infty(\Omega_d)$ such that

$$\begin{aligned} \alpha_k E'(u^k) + B^* \psi^k &= B^* \psi^{k-1} \\ Bu^k - (\nabla R)^{-1}(\psi^k) &= 0. \end{aligned} \tag{*}$$

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The Proximal Galerkin method consists of a finite element discretization of (*).

Proximal Galerkin for Signorini's Problem

For $\psi^0 = 0$, find $(u^k, \psi^k) \in V \times L^\infty(\Gamma_C)$ such that

$$\begin{aligned}\alpha_k(C\varepsilon(u^k), \varepsilon(v)) + (\psi^k, v \cdot \tilde{n})_{\Gamma_C} &= \alpha_k(f, v) + (\psi^{k-1}, v \cdot \tilde{n})_{\Gamma_C} \\ (u^k \cdot \tilde{n}, w)_{\Gamma_C} + (\exp(-\psi^k), w)_{\Gamma_C} &= (\phi, w)_{\Gamma_C}\end{aligned}$$

for all $(v, w) \in V \times L^\infty(\Gamma_C)$.

Proximal Galerkin for Signorini's Problem

For $\psi_h^0 = 0$, find $(u_h^k, \psi_h^k) \in V_h \times L^\infty(\Gamma_C)$ such that

$$\begin{aligned}\alpha_k(C\varepsilon(u_h^k), \varepsilon(v_h)) + (\psi_h^k, v_h \cdot \tilde{n})_{\Gamma_C} &= \alpha_k(f, v_h) + (\psi_h^{k-1}, v_h \cdot \tilde{n})_{\Gamma_C} \\ (u_h^k \cdot \tilde{n}, w_h)_{\Gamma_C} + (\exp(-\psi_h^k), w_h)_{\Gamma_C} &= (\phi, w_h)_{\Gamma_C}\end{aligned}$$

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Proximal Galerkin for Signorini's Problem

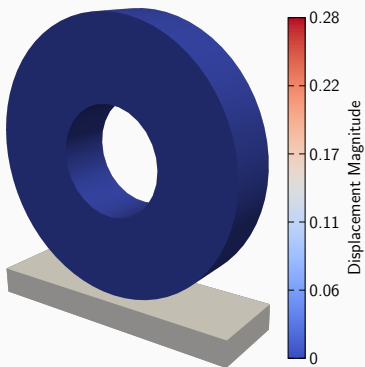
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for all $(v_h, w_h) \in V_h \times L^\infty(\Gamma_C)$. Here, we encode the constraint $v \cdot \tilde{n} \leq \phi$ with

$$R_x(a) = (\phi(x) - a) \ln(\phi(x) - a) - (\phi(x) - a).$$

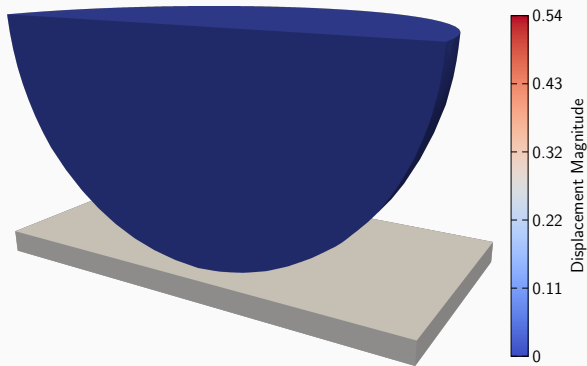
Signorini's Problem: Simulations



Deformation of a wheel with
 $f \equiv (0, 0, -1)$, $\alpha_k = 5$, $E = 2.5$, $\nu = 0.25$.

With help from: Yiming Yu, Brown University

Signorini's Problem: Simulations



Deformation of a hemisphere with
 $f \equiv (0, 0, -2)$, $\alpha_k = 5$, $E = 2.5$, $\nu = 0.25$.

With help from: Yiming Yu, Brown University

Reformulation

For $\psi_h^0 = 0$, find $(u_h^k, \kappa_h^k) \in V_h \times L^\infty(\Gamma_C)$ such that

$$\begin{aligned} (C\varepsilon(u_h^k), \varepsilon(v_h)) - (\kappa_h^k, v_h \cdot \tilde{n})_{\Gamma_C} &= (f, v_h) \\ (u_h^k \cdot \tilde{n}, w_h)_{\Gamma_C} + (\exp(-\psi_h^k), w_h)_{\Gamma_C} &= (\phi, w_h)_{\Gamma_C}. \end{aligned}$$

Here,

$$\kappa_h^k := \frac{1}{\alpha_k}(\psi_h^{k-1} - \psi_h^k).$$

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Remark: κ_h^k is an approximate Lagrange multiplier. $-\kappa_h^k$ represents the contact pressure.

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Remark: κ_h^k is an approximate Lagrange multiplier. $-\kappa_h^k$ represents the contact pressure.

Newton's Method

Find $\delta u_h, \delta \kappa_h$ such that

$$DF(u_h^k, \kappa_h^k)(\delta u_h, \delta \kappa_h) = -F(u_h^k, \kappa_h^k),$$

where

$$\langle F(u, \kappa), (v, w) \rangle = \begin{bmatrix} (C\varepsilon(u), \varepsilon(v)) - (\kappa, v \cdot \tilde{n})_{\Gamma_C} - (f, v) \\ (u \cdot \tilde{n}, w)_{\Gamma_C} + (\exp(-\psi^k) - \phi, w)_{\Gamma_C} \end{bmatrix},$$

$$\langle DF(u, \kappa)(\delta u, \delta \kappa), (v, w) \rangle = \begin{bmatrix} (C\varepsilon(\delta u), \varepsilon(v)) & -(\delta \kappa, v \cdot \tilde{n})_{\Gamma_C} \\ (\delta u \cdot \tilde{n}, w)_{\Gamma_C} & \alpha_k(\exp(-\psi^k)\delta \kappa, w)_{\Gamma_C} \end{bmatrix}.$$

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Preconditioning

$$\begin{bmatrix} (C\varepsilon(\delta u), \varepsilon(v)) & -(\delta\kappa, v \cdot \tilde{n})_{\Gamma_C} \\ (\delta u \cdot \tilde{n}, w)_{\Gamma_C} & \alpha_k(\exp(-\psi^k)\delta\kappa, w)_{\Gamma_C} \end{bmatrix} \rightarrow \begin{bmatrix} M & G^T \\ -G & K \end{bmatrix} =: J.$$

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We can block-LU factorize J as

$$J = LU = \begin{bmatrix} M & 0 \\ -G & S \end{bmatrix} \begin{bmatrix} I & M^{-1}G^T \\ 0 & I \end{bmatrix},$$

where $S = K - (-G)M^{-1}G^T = K + GM^{-1}G^T$ is the Schur complement of the block M .

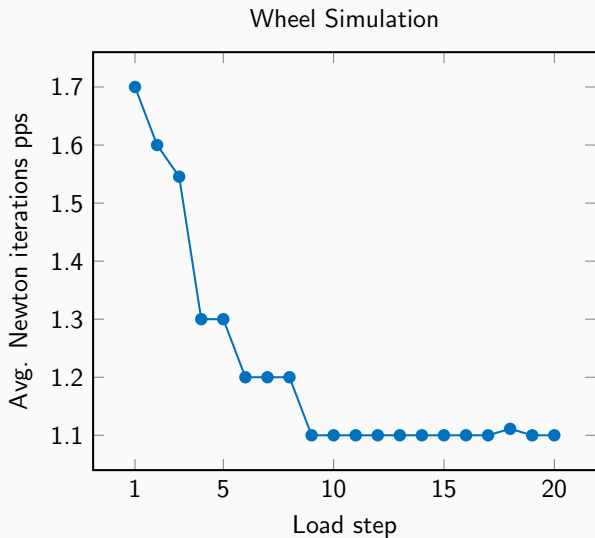
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where $S = K - (-G)M^{-1}G^T = K + GM^{-1}G^T$ is the Schur complement of the block M . Hence, we use MFEM's `BlockLowerTriangularPreconditioner`.

Newton Iterations



Multi-body Contact

Reformulation

We can rewrite the feasible set K as

$$K = \{v \in V \mid g(v) \geq 0 \text{ a.e. on } \Gamma_C\},$$

where $g(v) := \phi - v \cdot \tilde{n} : \Gamma_C \rightarrow \mathbb{R}$ is the *true gap*.

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How do we model multi-body contact?

Multi-body Contact

Given a true gap function g , we minimize

$$E : V \rightarrow \mathbb{R}, \quad E(u) = \sum_{i=1}^2 \frac{1}{2} \int_{\Omega_i} C^i \varepsilon(u^i) : \varepsilon(u^i) \, dx - \int_{\Omega_i} f^i \cdot u^i \, dx$$

over the feasible set

$$K = \{v = (v^1, v^2) \in V \mid g(v) \geq 0 \text{ a.e. on } \Gamma_C^1\},$$

where

$$V = \{v = (v^1, v^2) \in H^1(\Omega^1, \mathbb{R}^d) \times H^1(\Omega^2, \mathbb{R}^d) \mid \gamma v^i = g^i \text{ a.e. on } \Gamma_D^i\}.$$

Proximal Galerkin for Multi-body Contact

Define

$$a : V \times V \rightarrow \mathbb{R}, \quad a(u, v) = \sum_{i=1}^2 \int_{\Omega^i} C^i \varepsilon(u^i) : \varepsilon(v^i) \, dx$$

$$b : V \rightarrow \mathbb{R}, \quad b(v) = \sum_{i=1}^2 \int_{\Omega^i} f^i \cdot v^i \, dx.$$

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For $\psi_h^0 = 0$, find $(u_h^k, \kappa_h^k) \in V_h \times L^\infty(\Gamma_C^1)$ such that

$$\begin{aligned} a(u_h^k, v_h) + (\kappa_h^k, g'(u_h^k)v_h)_{\Gamma_C^1} &= b(v_h) \\ -(g(u_h^k), w_h)_{\Gamma_C^1} + (\exp(-\psi_h^k), w_h)_{\Gamma_C^1} &= 0 \end{aligned}$$

for all $(v_h, w_h) \in V_h \times L^\infty(\Gamma_C^1)$.

LLNL's Tribol is a “modular interface physics library featuring state-of-the-art contact physics methods” (Settgast et al. 2023).

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$$w \mapsto (g(u), w)_{\Gamma_c^1}, \quad (\delta u, w) \mapsto (g'(u)\delta u, w)_{\Gamma_c^1}$$

for a given displacement u .

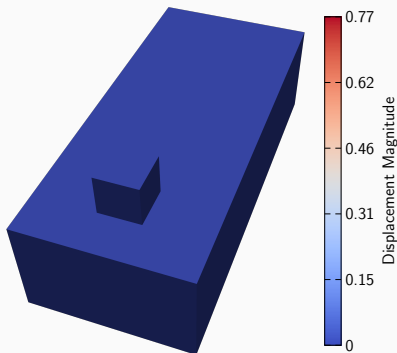
LLNL's Tribol is a “modular interface physics library featuring state-of-the-art contact physics methods” (Settgast et al. 2023). Tribol supplies the integrators

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for a given displacement u . We linearize the gap around a reference point at each load step,

$$g(u) := g(u_{\text{ref}}) + g'(u_{\text{ref}})(u - u_{\text{ref}}).$$

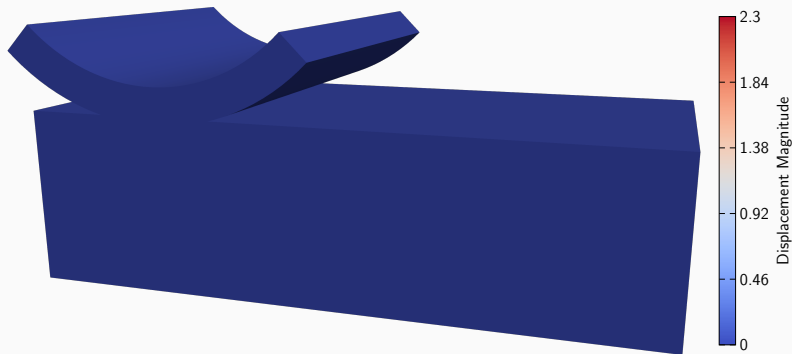
Multi-body Contact: Simulations



Deformation of a large rectangular block by a small cubic block, with $E_{\text{small-block}} = 1000$, $\nu_{\text{small-block}} = 0$,
 $E_{\text{large-block}} = 1$, $\nu_{\text{large-block}} = 0.499$.

With help from: Biswajit Khara, Vanderbilt University

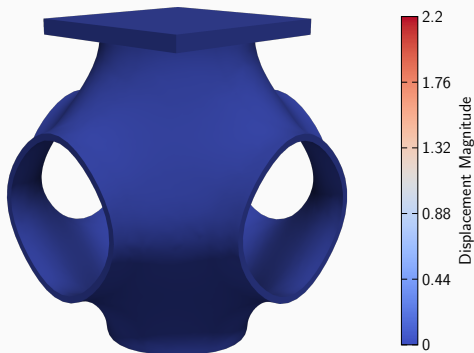
Multi-body Contact: Simulations



Deformation of a large rectangular block by an “iron”, with
 $E_{\text{iron}} = 1000$, $\nu_{\text{iron}} = 0$, $E_{\text{large-block}} = 1$, $\nu_{\text{large-block}} = 0.499$.

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Multi-body Contact: Simulations



Deformation of a triply periodic minimal surface by a plate,
with $E_{\text{plate}} = 1000$, $\nu_{\text{plate}} = 0$, $E_{\text{tpms}} = 1$, $\nu_{\text{tpms}} = 0.499$.

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Proximal Galerkin Features

Find $(u_h^k, \psi_h^k) \in V_h \times W_h$ such that

$$\begin{aligned}\alpha_k E'(u_h^k) + B^* \psi_h^k &= B^* \psi_h^{k-1} \\ Bu_h^k - (\nabla R)^{-1}(\psi_h^k) &= 0.\end{aligned}$$

Proximal Galerkin features...

- Feasible iterates,

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Can we avoid a saddle-point formulation?

A Primal Formulation

First order optimality yields: Find $u^k \in V$ such that

$$\begin{aligned}\alpha_k(C\varepsilon(u^k), \varepsilon(v)) - (\ln(\phi - u^k \cdot \tilde{n}), v \cdot \tilde{n})_{\Gamma_C} \\ = \alpha_k(f, v) - (\ln(\phi - u^{k-1} \cdot \tilde{n}), v \cdot \tilde{n})_{\Gamma_C}\end{aligned}$$

for all $v \in V_0$.

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for all $v_h \in V_h$. Can we immediately discretize?

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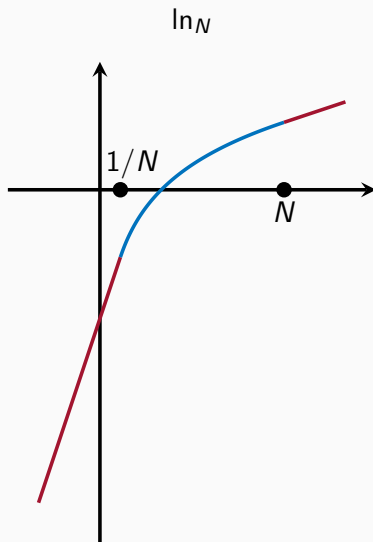
for all $v \in V_0$. Can we immediately discretize?

Legendre Function Regularization

Since $\ln$ is singular, we regularize it through a *first order Taylor approximation* centered at $1/N$ and N for given $N > 1$.

$$\ln_N(a) := \begin{cases} \ln(a) & \text{if } a \in [1/N, N], \\ \ln(1/N) + N(a - 1/N) & \text{if } a < 1/N, \\ \ln(N) + \frac{1}{N}(a - N) & \text{if } a > N. \end{cases}$$

Legendre Function Regularization

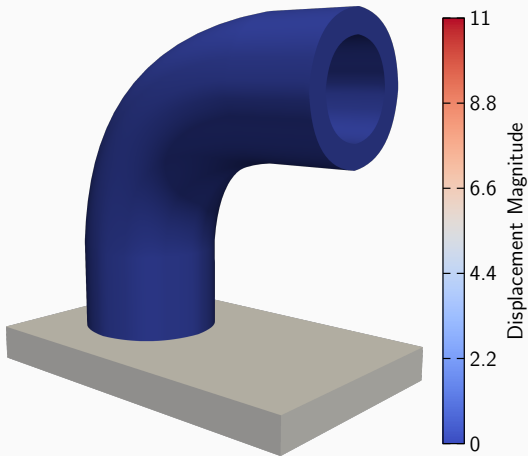


We thus solve: Find $u^k \in V$ such that

$$\begin{aligned}\alpha_k(C\varepsilon(u^k), \varepsilon(v)) - (\textcolor{red}{ln}_N(\phi - u^k \cdot \tilde{n}), v \cdot \tilde{n})_{\Gamma_C} \\ = \alpha_k(f, v) - (\textcolor{red}{ln}_N(\phi - u^{k-1} \cdot \tilde{n}), v \cdot \tilde{n})_{\Gamma_C}\end{aligned}$$

for all $v \in H_0^1(\Omega, \mathbb{R}^3)$.

Signorini's Problem: Simulations



Deformation of a 90-degree pipe elbow with
 $N \equiv 10^6, \alpha_k = 2k, E = 2.5, \nu = 0.25$.

Conclusion

We have presented *proximal Galerkin*, a structure-preserving, mesh independent numerical method, and a regularized-primal form, for contact.

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- Tresca friction.
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- A priori error analysis.

CIGMO

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GEOMETRIC MECHANICS
AND OPTIMIZATION**



National Nuclear Security Administration

Thank you!

-  Chen, Gong and Marc Teboulle (1993). “**Convergence Analysis of a Proximal-Like Minimization Algorithm Using Bregman Functions**”. In: *SIAM Journal on Optimization* 3.3, pp. 538–543. DOI: 10.1137/0803026.
-  Settghost, Randolph R et al. (Mar. 2023). ***Tribol Interface Physics Library***. DOI: 10.11578/dc.20230623.4. URL: <https://www.osti.gov/biblio/code-108756>.